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Abstract. Let Γ be the fundamental group of a compact riemannian manifold X of sectional curvature $K \leq -1$ and dimension $n \geq 3$. We suppose that $\Gamma = A *_C B$ is the free product of its subgroups A and B over the amalgamated subgroup C. We prove that the critical exponent $\delta(C)$ of C satisfies $\delta(C) \geq n-2$. The equality happens if and only if there exist an embedded compact hypersurface $Y \subset X$, totally geodesic, of constant sectional curvature -1, whose fundamental group is C and which separates X in two connected components whose fundamental groups are A and B. Similar results hold if Γ is an HNN extension, or more generally if Γ acts on a simplicial tree without fixed point.

1. Introduction

In [15], Y. Shalom proved the following theorem which says that for every lattice Γ in the hyperbolic space and for any decomposition of Γ as an amalgamated product $\Gamma = A *_C B$, the group C has to be "big". In order to measure how "big" C is, let us define the critical exponent of a discrete group C acting on a Cartan Hadamard manifold by

$$\delta(C) = inf\{s > 0 \quad | \quad \Sigma_{\gamma\in\Gamma} e^{-sd(\gamma x,x)} < +\infty\}.$$

Theorem 1.1. *Let Γ be a lattice in $PO(n,1)$. Assume that Γ is an amalgamated product of its subgroups A and B over C. Then, the critical exponent $\delta(C)$ of C satisfy $\delta(C) \geq n-2$.*

An example is given by any n-dimensional hyperbolic manifold X which contains a compact separating connected totally geodesic hypersurface Y. The Van Kampen theorem then says that the fundamental group Γ of X is isomorphic to the free product of the fundamental groups of the two halves of $X - Y$ amalgamated over the fundamental group C of the incompressible hypersurface Y. Such examples do exist in dimension 3 thanks to the W.Thurston's hyperbolization theorem. In any dimension, A. Lubotsky showed that any standard arithmetic lattice of $PO(n,1)$ has a finite cover whose fundamental group is an amalgamated product, cf. [11]. In fact, A. Lubotsky proved that any standard arithmetic lattice Γ has a finite index subgroup Γ_0 which is mapped onto a nonabelian free group. A nonabelian free group can be written in infinitely many ways as an amalgamated product, so one get infinitely many decomposition of Γ_0 as an amalgamated

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product by pulling back the amalgamated decomposition of the nonabelian free group.

In these cases there is equality in theorem 1.1, ie. $\delta(C) = n - 2$ where C is the fundamental group of Y, and Y. Shalom suggested in [15] that the equality case in the theorem 1.1 happens only in that case.

The aim of this paper is to show that the theorem 1.1 still holds when Γ is the fundamental group of a compact riemannian manifold of variable sectional curvature less than or equal to -1, and characterize the equality case.

Theorem 1.2. *Let X be an n-dimensional compact riemanniann manifold of sectional curvature $K \leq -1$. We assume that the fundamental group Γ of X is an amalgamated product of its subgroups A and B over C and that neither A nor B equals Γ. Then, the critical exponent $\delta(C)$ of C satisfy $\delta(C) \geq n-2$. Equality $\delta(C) = n-2$ happens if and only if C cocompactly preserves a totally geodesic isometrically embedded copy $\mathbb{H}^{n-1}$ of the hyperbolic space of dimension $n-1$. Moreover, in the equality case, the hypersurface $Y^{n-1} := \mathbb{H}^{n-1}/C$ is embedded in X and separates X in two connected components whose fundamental groups are respectively A and B.*

Remark 1.3. *(i) By the assumption on A or B not being equal to Γ we exclude the trivial decomposition $\Gamma = \Gamma *_C C$ where $A = \Gamma$ and $C = B$ can be an arbitrary subgroup of Γ, for example any cyclic subgroup, in which case the conclusion of theorem 1.2 fails. Also note that because of this assumption on A and B, we have $A \neq C$ and $B \neq C$.*

(ii) Let us recall that standard arithmetic lattices in $PO(n,1)$ have finite index subgroup with infinitely many non equivalent decompositions as amalgamated products, cf. [11]. *In fact, among these decompositions, all but finitely many of them are such that $\delta(C) > n-2$. Indeed, by theorem 1.2, if $\delta(C) = n-2$ then C is the fundamental group of an embedded totally geodesic hypersurface in X, but there are only finitely many totally geodesic hypersurfaces by* [21].

When a group is an amalgamated product, it acts on a simplicial tree without fixed point and theorem 1.1 is a particular case of the

Theorem 1.4. *(*[15]*, theorem 1.6). Let $\Gamma \subset SO(n,1)$, $n \geq 3$, be a lattice Suppose Γ acts on a simplicial tree T without fixed vertex. Then there is an edge of T whose stabilizer C satisfies $\delta(C) \geq n-2$.*

In the case Γ is cocompact, the conclusion of theorem 1.4 holds for the stabilizer of any edge which separates the tree T in two unbounded components, and the proof of this is exactly the same as the proof of theorem 1.2. In particular, when the action of Γ on T is minimal, (ie. there is no proper subtree of T invariant by Γ), the conclusion of theorem 1.4 holds for every edge of T, in the variable curvature setting, and we are able to handle the equality case.

Theorem 1.5. *Let Γ be the fundamental group of an n-dimensional compact riemannian manifold X of sectional curvature less than or equal to -1. Suppose Γ acts minimally on a simplicial tree T without fixed point. Then, the stabilizer C of every edge of T satisfies $\delta(C) \geq n-2$. The equality $\delta(C) = n-2$ happens if and only if there exist a compact totally geodesic hypersurface $Y \subset X$ with fundamental group $\pi_1(Y) = C$. Moreover, in that case, Y with its induced metric has constant sectional curvature -1.*

Another interesting case contained in theorem 1.5 is the case of HNN extension. Let us recall the definition of an HNN extension. Let A and C be groups and $f_1 : C \to A$, $f_2 : C \to A$ two injective morphisms of C into A. The HNN extension $A*_C$ is the group generated by A and an element t with the relations $tf_1(\gamma)t^{-1} = f_2(\gamma)$. For example, let X be a compact manifold containing a non separating compact incompressible hypersurface $Y \subset X$. Let A be the fundamental group of the manifold with boundary $X-Y$ obtained by cuting X along Y and let C be the fundamental group of Y. The boundary of $X-Y$ consists in two connected components $Y_1 \subset X-Y$ and $Y_2 \subset X-Y$ homeomorphic to Y. By the incompressibility assumption, these inclusions give rise to two embeddings of C into A, and the fundamental group of X is the associated HNN extension $A*_C$.

Theorem 1.6. *Let Γ be the fundamental group of a compact riemannian manifold X of dimension n and sectional curvature less than or equal to -1. Suppose that $\Gamma = A*_C$ where A is a proper subgroup of Γ. Then, we have $\delta(C) \geq n-2$ and equality $\delta(C) = n-2$ if and only if there exist a non separating compact totally geodesic hypersurface $Y \subset X$ with fundamental group $\pi_1(Y) = C$. Moreover, in that case, Y with its induced metric is of constant sectional curvature -1, and the HNN decomposition arising from Y is the one we started with.*

Let us summarize the ideas of the proof of theorem 1.2. We work on $\tilde{X}/C$. The amalgamation assumption provides an essential hypersurface Z in $\tilde{X}/C$, namely Z is homologically non trivial in $\tilde{X}/C$. The volume of all hypersurfaces homologous to Z is bounded below by a positive constant because their systole are bounded away from zero. We then construct a smooth map $F : \tilde{X}/C \to \tilde{X}/C$, homotopic to the identity which contracts the volume of all compact hypersurfaces Y by the factor $\left(\frac{\delta(C)}{n-2}\right)^{n-1}$, namely $vol_{n-1}F(Y) \leq \left(\frac{\delta(C)}{n-2}\right)^{n-1} vol_{n-1}Y$. This contracting property together with the lower bound of the volume of hypersurfaces in the homology class of Z gives the inequality $\delta(C) \geq n-2$. This map is different from the map constructed in [3], in particular it can be defined under the single condition that the limit set of C is not reduced to one point. Moreover, its derivative has an upper bound depending only on the critical exponent of C.

The equality case goes as follows. When $\delta(C) = n-2$, the map $F : \tilde{X}/C \to \tilde{X}/C$ contracts the $(n-1)$-dimensional volumes, ie. $|Jac_{n-1}F| \leq 1$.

This contracting property is infinitesimally rigid in the following sense. Let us consider a lift $\tilde{F}$ of F. If $|Jac_{n-1}\tilde{F}(x)| = 1$ at some point $x \in \tilde{X}$, then $\tilde{F}(x) = x$, there exists a tangent hyperplane $E \subset T_x\tilde{X}$ such that $D\tilde{F}(x)$ is the orthogonal projector of $T_x\tilde{X}$ onto E and the limit set Λ_C is contained in the *topological equator* $E(\infty) \subset \partial\tilde{X}$ associated to E. By topological equator $E(\infty) \subset \partial\tilde{X}$ associated to E, we mean the set of end points of those geodesic rays starting at x tangently to E.

We then prove the existence of a point $x \in \tilde{X}$ such that

$$|Jac_{n-1}\tilde{F}(x)| = 1. \tag{1.1}$$

If there would exist a minimizing cycle in the homology class of Z in $\tilde{X}/C$, any point of such a cycle would satisfy (1.1). As no such minimizing cycle a priori exists because of non compactness of $\tilde{X}/C$, we prove instead the existence of a L^2 harmonic $(n-1)$-form dual to Z, which is enough to prove existence of a point x such that (1.1) holds.

At this stage of the proof, there is a big difference between the constant curvature case and the variable curvature case.

In the constant curvature case, any topological equator bounds a totally geodesic hyperbolic hypersurface $\mathbb{H}^{n-1}$, and therefore, as the group C preserves $\Lambda_C \subset E(\infty) = \partial\mathbb{H}^{n-1}$, it is not hard to see that C also preserves $\mathbb{H}^{n-1}$ and acts cocompactly on it, and the hypersurface of the equality case in theorem 1.2 is $\mathbb{H}^{n-1}/C$, [2].

In the variable curvature case, we first show the existence of a C-invariant totally geodesic hypersurface $\tilde{Z}_\infty \subset \tilde{X}$ whose boundary at infinity coincides with $\Lambda(C)$, and then we show that $\tilde{Z}_\infty$ is isometric to the real hyperbolic space. We then show that $Y =: \mathbb{H}^{n-1}_{\mathbb{R}}/C$, which is compact, injects in $X = \tilde{X}/\Gamma$ and separates X in two connected components whose fundamental groups are A and B respectively.

In order to show the existence of such a totally geodesic hypersurface $\tilde{Z}_\infty$, we first prove that C is a convex cocompact group, ie. the convex hull of the limit set of C in $\tilde{X}$ has a compact quotient under the acton of C, and that the limit set of C is homeomorphic to an $(n-2)$-dimensional sphere.

The convex cocompactness property of C and the fact that the limit set $\Lambda(C)$ of C is homeomorphic to an $(n-2)$-dimensional topological sphere are the two key points in the equality case.

This compactness property then allows us to prove the existence of a mimimizing current in the homology class of the essential hypersurface $Z \subset \tilde{X}/C$. By regularity theorem this minimizing current $\tilde{Z}_\infty$ is a smooth manifold except at a singular set of codimension at least 8. By the contracting properties of our map F, $\tilde{Z}_\infty$ is fixed by F and the geometric properties of F at fixed points where the $(n-1)$-jacobian of F equals 1 allows us to prove that $\tilde{Z}_\infty$ is totally geodesic and isometric to the hyperbolic space.

Let us now briefly describe the proof of the convex cocompactness property of C in the equality case.

The group C (or a finite index subgroup of it) actually globally preserves a smooth cocompact hypersurface $\tilde{Z} \subset \tilde{X}$ which separates $\tilde{X}$ into two connected components and whose boundary $\partial\tilde{Z} \subset \partial\tilde{X}$ coincides with $\Lambda_C \subset E(\infty)$. In the case where C wouldn't be convex cocompact, we are able to find an horoball $HB(\theta_0)$ centered at some point $\theta_0 \in \Lambda_C$ in the complementary of which lies the hypersurface $\tilde{Z}$.

The contradiction then comes from the following.

Consider a sequence of points $\theta_i \in \partial\tilde{X}$ converging to θ_0 and geodesic rays α_i starting from the point $x \in \tilde{X}$ at which $|Jac_{n-1}(x)| = 1$ and ending up at θ_i. These geodesic rays have to cross $\tilde{Z}$ at points z_i which are at bounded distance from the orbit Cx of x, therefore the shadows $\mathcal{O}_i$ of balls centered at these z_i enlighted from x have to contain points of Λ_C by the shadow lemma of D. Sullivan. On the other hand, we show that it is possible to choose the sequence θ_i in such a way that these shadows $\mathcal{O}_i$ don't meet Λ_C. This property $\mathcal{O}_i \cap \Lambda_C = \emptyset$ comes from a choice of θ_i such that the distance between z_i and the set H of all geodesics rays at x tangent to $E \subset T_x\tilde{X}$ tends to ∞. Intuitively, in order to chose z_i as far as possible from H, the points θ_i have to be chosen tranversally to Λ_C. This transversality condition is not well defined because the limit set Λ_C might be highly non regular. Thus, in order to prove that such a choice is possible, we argue again by contradiction. If for any choice of a sequence θ_i converging to θ_0, the distance between z_i and H stays bounded, then the Gromov distances $d(\theta_i, \theta_0)$ between θ_i and θ_0 satisfy $d(\theta_i, \Lambda_C) = o(d(\theta_i, \theta_0))$, and therefore any tangent cone of Λ_C at θ_0 would coincide with a tangent cone of $\partial\tilde{X}$ at θ_0, which is known to be topologically $\mathbb{R}^{n-1}$. But on the other hand, the existence of a point x such that $|Jac_{n-1}(x)| = 1$ and the fact that C acts uniformly quasiconformally with respect to the Gromov distance on $\partial\tilde{X}$ imply that the Alexandroff compactification of the above tangent cone of Λ_C at θ_0 is homeomorphic to Λ_C which is contained in a topological sphere S^{n-2}, leading to a contradiction.

From convex cocompactness of C and the fact that the limit set of C is a topological $(n-2)$-dimensional sphere, there is an alternative proof of the existence of a totally geodesic C-invariant copy of the hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}} \subset \tilde{X}$ which consists in observing that the topological dimension and the Hausdorff dimension of the limit set $\Lambda(C)$ are equal to $n-2$ and then use the following result of M. Bonk and B. Kleiner (which we quote in the riemannian manifold setting although it remains true for $CAT(-1)$ spaces) instead of the (simpler) minimal current argument.

Theorem 1.7. [4] *Let X be a Cartan Hadamard n-dimensional manifold whose sectional curvature satisfy $K \leq -1$, and C a convex cocompact discrete subgroup of isometries of X with limit set Λ_C. Let us assume that the topological dimension and the Hausdorff dimension (with respect to the Gromov distance on $\partial\tilde{X}$) of Λ_C coincide and are equal to an integer p. Then, C*

preseves a totally geodesic embedded copy of the real hyperbolic space $\mathbb{H}^{p+1}$*, with* $\partial\mathbb{H}^{p+1} = \Lambda_C$*.*

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2. Essential hypersurfaces

Let Γ be a discrete cocompact group of isometries of a n-dimensional Cartan-Hadamard manifold $(\tilde{X}, \tilde{g})$ whose sectional curvature satisfies $K_{\tilde{g}} \leq -1$. Let us assume that the compact manifold $X = \tilde{X}/\Gamma$ is orientable. Let us also assume that $\Gamma = A *_C B$ is an amalgamated product of its subgroups A and B over C.

We first reduce to the case where $[\Gamma : C]$ is infinite.

Namely, if $[\Gamma : C] < \infty$, then the critical exponent $\delta(C) = \delta(\Gamma) \geq n - 1$, and the equality in theorem 1.2 holds.

We then can assume that $[\Gamma : C] = \infty$.

Lemma 2.1. *Let* $\Gamma = A *_C B$ *be as above, with* $[\Gamma : C] = \infty$*. If neither* A *nor* B *equals* Γ*, then* $H_{n-1}(\tilde{X}/C, \mathbb{Z}) \neq 0$*.*

Proof : The Mayer-Vietoris sequence coming from the decomposition $\Gamma = A *_C B$ writes cf. [6], Corollary 7.7,

$$H_n(\tilde{X}/C, \mathbb{Z}) \to H_n(\tilde{X}/A, \mathbb{Z}) \oplus H_n(\tilde{X}/B, \mathbb{Z}) \to H_n(\tilde{X}/\Gamma, \mathbb{Z}) \to ...$$

$$... \to H_{n-1}(\tilde{X}/C, \mathbb{Z}) \to ...$$

As $[\Gamma : C] = \infty$, $H_n(\tilde{X}/C, \mathbb{Z}) = 0$ thus, if $H_{n-1}(\tilde{X}/C, \mathbb{Z}) = 0$, we deduce from the Mayer-Vietoris sequence that $H_n(\tilde{X}/A, \mathbb{Z}) \oplus H_n(\tilde{X}/B, \mathbb{Z})$ is isomorphic to $H_n(\tilde{X}/\Gamma, \mathbb{Z})$. As $H_{n-1}(\tilde{X}/\Gamma, \mathbb{Z}) = \mathbb{Z}$, we then deduce that either $[\Gamma : A] = \infty$ and $B = \Gamma$, or $[\Gamma : B] = \infty$ and $A = \Gamma$. $\square$

In fact in the sequel of the paper we will make use of a smooth essential hypersurface Z in $\tilde{X}/C$.

Definition 2.2. *A compact smooth orientable hypersurface* Z *of an* n*-dimensional manifold* Y *is essential in* Y *if* $i_*([Z]) \neq 0$ *where* $[Z] \in H_{n-1}(Z, \mathbb{R})$ *denotes the fundamental class of* Z *and* $i_* : H_{n-1}(Z, \mathbb{R}) \to H_{n-1}(Y, \mathbb{R})$ *the morphism induced by the inclusion* $i : Z \hookrightarrow Y$*.*

The end of this section is devoted to finding such an hypersurface Z in $\tilde{X}/C$.

Let us recall a few facts about amalgamated products and their actions on trees, following [14]. Let $\Gamma = A *_C B$ be an amalgamated products of its subgroups A and B over C. Then, Γ acts on a simplicial tree $\tilde{T}$ with a fundamental domain $T \subset \tilde{T}$ being a segment, ie. an edge joining two vertices. Let us describe this tree $\tilde{T}$. There are two orbits of vertices Γv_A and Γv_B, the stabilizer of the edge v_A (resp. v_B) being A,(resp. B). There is one orbit of edges Γe_C, the stabilizer of the edge e_C being C. The fundamental domain T can be chosen as the edge e_C joining the two vertices v_A and v_B. The set of vertices adjacent to v_A, (resp. v_B), is in one to one correspondance with A/C, (resp. B/C). Note that as neither A nor B are equal to Γ, then $[A : C] \neq 1$ and $[B : C] \neq 1$, therefore for an arbitrary point t_0 on the edge e_C we see that $\tilde{T} - t_0$ is a disjoint union of two unbounded connected components. This fact will be used later on.

Let us consider a continuous Γ-equivariant map $\tilde{f} : \tilde{X} \to T$ where T is the Bass-Serre tree associated to the amalgamation $\Gamma = A *_C B$. One regularizes $\tilde{f}$ such that it is smooth in restriction to the complementary of the inverse image of the set of vertices of T. Let t_0 a regular value of $\tilde{f}$ contained in that edge of T which is fixed by the subgroup C and define $\tilde{Z} = \tilde{f}^{-1}(t_0)$. $\tilde{Z}$ is a smooth orientable possibly not connected hypersurface in $\tilde{X}$, globally C-invariant. Let us write $Z = \tilde{Z}/C$. We will show $Z \subset \tilde{X}/C$ is compact and that one of the connected components of Z is essential.

Lemma 2.3. *$Z \subset \tilde{X}/C$ is compact.*

Proof : Let us show that for any sequence $z_n \in \tilde{Z}$, there exists a subsequence z_{n_k} and $\gamma_k \in C$ such that $\gamma_k z_{n_k}$ converges. As Γ is cocompact, there exists $g_n \in \Gamma$ such that the set $(g_n z_n)$ is relatively compact. Let $g_{n_k} z_{n_k}$ a subsequence which converges to a point $z \in \tilde{X}$. By continuity, the sequence $\tilde{f}(g_{n_k} z_{n_k})$ converges to $\tilde{f}(z)$, and by equivariance we get

$$g_{n_k} \tilde{f}(z_{n_k}) = g_{n_k} t_0 \to \tilde{f}(z)$$

when k tends to ∞. As Γ acts in a simplicial way on the tree T and transitively on the set of edges, the sequence $g_{n_k} t_0$ is stationary, ie $g_{n_k} t_0 = t_0' = g t_0$ for k large enough. Thus $g^{-1} g_{n_k} = \gamma_k \in C$ for k large enough since it fixes t_0 and $\gamma_k z_{n_k} = g^{-1} g_{n_k} z_{n_k}$ converges to $g^{-1}(z)$ $\square$

The smooth compact hypersurface Z we constructed might be not connected. Let us write $Z = Z_1 \cup Z_2 \cup ... \cup Z_k$ where the Z_j's are the connected components of Z. Each Z_j is a compact smooth oriented hypersurface of $\tilde{X}/C$.

The aim of what follows is to prove that at least one component Z_i of Z is essential.

Lemma 2.4. *There exists $i \in [1, k]$ such that Z_i is essential in $\tilde{X}/C$.*

Proof : If there exists a Z_i which doesn't separate $\tilde{X}/C$ in two connected components, then Z_i is essential in $\tilde{X}/C$. So we can assume that every Z_j, $j = 1, ...k$, does separate $\tilde{X}/C$ in two connected components. In that case we will show that there exists a Z_i which separates $\tilde{X}/C$ in two unbounded connected components which easily implies that Z_i is essential.

Let us denote U_l, $l = 1, 2, ..., p$, the connected components of $\tilde{X}/C - \cup_{j=1}^{k} Z_j$.

Claim : at least two components U_m, $U_{m'}$ are unbounded.

Assuming the claim let us finish the proof of the lemma. For each Z_j we denote V_j, V'_j the two connected components of $\tilde{X}/C - Z_j$. Then $U_m = W_1 \cap W_2 \cap ... \cap W_k$ where for each j, $W_j = V_j$ or $W_j = V'_j$. In the same way, $U_{m'} = W'_1 \cap W'_2 \cap ... \cap W'_k$. As $U_m \cap U_{m'} = \emptyset$, there exists $i \in [1, k]$ such that $W_i \cap W'_i = \emptyset$, thus $U_m \subset V_i$ and $U_{m'} \subset V'_i$ or $U_m \subset V'_i$ and $U_{m'} \subset V_i$ so Z_i separates $\tilde{X}/C$ into two unbounded components. This proves the lemma.

Let us prove the claim.

We have already noticed that $T - \{t_0\}$ is the disjoint union of two unbounded connected components T_1 and T_2. As C acts on T isometrically and simplicially then $T/C - \{t_0\} = T_1/C \cup T_2/C$ is the disjoint union of two unbounded connected components. Let $\bar{f} : \tilde{X}/C \to T/C$ the quotient map of $\tilde{f}$. For each component U_i, we have

$\bar{f}(U_i) \subset T_1/C$ or $\bar{f}(U_i) \subset T_2/C$, thus we can conclude the claim because $\bar{f}$ is onto $\square$

Let $\pi : \tilde{X} \to \tilde{X}/C$ be the natural projection. For any $i = 1, 2, ..., k$, let us denote $\{\tilde{Z}_i^j\}_{j \in J}$ the set of connected components of $\tilde{Z}_i =: \pi^{-1}(Z_i)$.

For each $i \in [1, k]$, we claim that C acts transitively on the set $\{\tilde{Z}_i^j\}_{j \in J}$. Namely, let us consider $\tilde{Z}_i^j$ $\tilde{Z}_i^{j'}$, $\tilde{z} \in \tilde{Z}_i^j$, $\tilde{z}' \in \tilde{Z}_i^{j'}$, and write $z = \pi\tilde{z} \in Z_i$ and $z' = \pi\tilde{z}' \in Z_i$. Let α be a continuous path on Z_i such that $\alpha(0) = z$ and $\alpha(1) = z'$, and $\tilde{\alpha}$ the lift of α such that $\tilde{\alpha}(0) = \tilde{z}$. We have $\pi\tilde{\alpha}(1) = z'$ and $\tilde{\alpha}(1) \in \tilde{Z}_i^j$ for some j, thus, there exists $c \in C$ such that $c(\tilde{\alpha}(1)) = \tilde{z}'$ and therefore $c\tilde{Z}_i^j = \tilde{Z}_i^{j'}$.$\square$

Let us denote C_i^j the stabilizer of $\tilde{Z}_i^j$, and $Z_i^j = \tilde{Z}_i^j / C_i^j \subset \tilde{X}/C_i^j$. Let us write $p : \tilde{X}/C_i^j \to \tilde{X}/C$ the natural projection.

Lemma 2.5. *The restriction of p to Z_i^j is a diffeomorphism onto Z_i. In particular, Z_i^j is compact.*

proof : Let z and z' be two points in Z_i^j such that $p(z) = p(z')$. Let $\tilde{z}$ and $\tilde{z'}$ be lifts of z and z' in $\tilde{X}$. These two points $\tilde{z}$ and $\tilde{z'}$ which are in $\tilde{Z}_i$ actually belong to the same connected component $\tilde{Z}_i^j$ because for $j \neq j'$, $\tilde{Z}_i^{j'}/C_i^j \cap \tilde{Z}_i^j/C_i^j = \emptyset$. As $p(z) = p(z')$, there exits $c \in C$ such that $\tilde{z}' = c\tilde{z}$, thus $c \in C_i^j$, and $z = z'$, therefore the restriction of p to Z_i^j is injective. The surjectivity comes from the fact that $\pi^{-1} Z_i = \bigcup_{j \in J} \tilde{Z}_i^j$ and C acts transitively on the set $\{\tilde{Z}_i^j\}_{j \in J}$. $\square$

Let us consider the integer $i \in [1, k]$ as in lemma 2.4, ie. such that $Z_i \hookrightarrow \tilde{X}/C$ is essential, and choose $\tilde{Z}_i^l$ one component of $\pi^{-1}(Z_i)$.

After possibly replacing C_i^l by an index two subgroup, we may assume that C_i^l globally preserves each of the two connected components U_i^l and V_i^l of $\tilde{X} - \tilde{Z}_i^l$.

Lemma 2.6. *Let i, l and C_i^l be chosen as above. The compact hypersurface $Z_i^l = \tilde{Z}_i^l/C_i^l$ is essential in $\tilde{X}/C_i^l$. Moreover the two connected components U_i^l/C_i^l and V_i^l/C_i^l of $\tilde{X}/C_i^l - Z_i^l$ are unbounded.*

Proof : Let us consider $p : \tilde{X}/C_i^l \to \tilde{X}/C$. By lemma 2.5, the restriction of p to Z_i^l is a diffeomorphism onto Z_i, therefore Z_i^l is essential in $\tilde{X}/C_i^l$ because Z_i is essential in $\tilde{X}/C$. As C_i^l preseves U_i^l and V_i^l, Z_i^l separates $\tilde{X}/C_i^l$ into two connected components U_i^l/C_i^l and V_i^l/C_i^l. and as Z_i^l is essential in $\tilde{X}/C_i^l$, U_i^l/C_i^l and V_i^l/C_i^l are unbounded. □

In the sequel of the paper we will denote $\tilde{Z}' = \tilde{Z}_i^l$, $C' = C_i^l$ and $Z' = Z_i^l = \tilde{Z}_i^l/C_i^l$.

3. Isosystolic inequality

In this section we summarize facts and results due to M.Gromov, [9]. Let Z be a p-dimensional compact orientable manifold and $i : Z \hookrightarrow Y$ an embedding of Z into Y where Y is an aspherical space. We suppose that $i_*([Z]) \neq 0$ where $i_* : H_p(Z, \mathbb{R}) \to H_p(Y, \mathbb{R})$ is the morphism induced by the embedding $Z \hookrightarrow Y$. Let us fix a riemanniann metric g on Z. For each $z \in Z$ we consider the set $\mathcal{C}_z$ of those loops α at z such that $i \circ \alpha$ is homotopically non trivial in Y.

Let us define the systole of (Z, g, i) at the point z by

Definition 3.1. $sys_i(Z, g, z) = inf\{lengh(\alpha),\ \alpha \in \mathcal{C}_z\}$

and the systole of (Z, g, i) by

Definition 3.2. $sys_i(Z, g) = inf\{sys_i(Z, g, z)\ z \in Z\}$.

The following isosystolic inequality, due to M.Gromov says that the volume of any essential submanifold Z of an aspherical space Y relatively to any riemanniann metric on Z is universally bounded below by it's systole.

Theorem 3.3. [9] *There exists a constant C_p such that for each p-dimensional riemanniann manifold (Z, g) and any embedding $Z \hookrightarrow Y$ into an aspherical space Y such that $i_*([Z]) \neq 0$ where $i_* : H_p(Z, \mathbb{R}) \to H_p(Y, \mathbb{R})$ is the induced morphism in homology, then $vol_p(Z, g) \geq C_p(sys_i(Z, g))^p$*

We will apply this volume estimates to the essential hypersurface $i : Z \hookrightarrow \tilde{X}/C$ that we constructed in lemma 2.6.

The following lemma is immediate.

Lemma 3.4. *Let C be a discrete group acting on a simply connected manifold $\tilde{X}$, $\tilde{Z}$ a C-invariant hypersurface of $\tilde{X}$ and $i : Z = \tilde{Z}/C \hookrightarrow \tilde{X}/C$ the natural inclusion. Let g any riemanniann metric on Z and $\tilde{g}$ the lift of g to $\tilde{Z}$. Then, for any $z \in Z$ we have,*

$$sys_i(Z, g, z) = inf\{d_{\tilde{g}}(\tilde{z}, \gamma\tilde{z}), \gamma \in C\}$$

where $\tilde{z} \in \tilde{Z}$ is a lift of $z \in Z$ and $d_{\tilde{g}}$ is the distance induced by $\tilde{g}$ on $\tilde{Z}$.

Proof : Let $\alpha \in \mathcal{C}_z$ a loop based at $z \in Z$. As $i \circ \alpha$ is an homotopically non trivial loop at $i(z) = z$ in $\tilde{X}/C$, its lift $\widetilde{i \circ \alpha}$ at some $\tilde{z} \in \tilde{Z}$ ends up at $\gamma\tilde{z}$ for some $\gamma \in C$.□

4. Volume of hypersurfaces in $\tilde{X}/C$

Let $(\tilde{X}, \tilde{g})$ be a n-dimensional Cartan-Hadamard manifold whose sectional curvature satisfies $K_{\tilde{g}} \leq -1$ and C a discrete group of isometries of $(\tilde{X}, \tilde{g})$. We assume that the group C is non elementary, namely C fixes neither one nor two points in the geometric boundary $\partial\tilde{X}$ of $(\tilde{X}, \tilde{g})$.

The aim of this section is to construct a map $F : \tilde{X}/C \to \tilde{X}/C$ such that for any compact hypersurface Z of $\tilde{X}/C$, we have

$$vol_{n-1}(F(Z)) \leq \Big(\frac{\delta + 1}{n - 1}\Big)^{n-1} vol_{n-1}(Z)$$

where δ is the critical exponent of C and $vol_{n-1}(Z)$ stands for the $(n-1)$-dimensional volume of the metric on Z induced from g. For every subgroup $C' \subset C$ and any hypersurface Z' of $\tilde{X}/C'$ the lift $F' : \tilde{X}/C' \to \tilde{X}/C'$ of F will also verify

$$vol_{n-1}(F'(Z')) \leq \Big(\frac{\delta + 1}{n - 1}\Big)^{n-1} vol_{n-1}(Z').$$

In order to construct the map F we need a few prelimiraries. We consider a finite positive Borel measure μ on the boundary $\partial\tilde{X}$ whose support contains at least two points. Let us fix an origin $o \in \tilde{X}$ and denote $B(x, \theta)$ the Busemann function defined for each $x \in \tilde{X}$ and $\theta \in \partial\tilde{X}$ by

$$B(x, \theta) = \lim_{t \to \infty} dist(x, c(t)) - t$$

where $c(t)$ is the geodesic ray such that $c(0) = o$ and $c(+\infty) = \theta$.

Let $\mathcal{D}_\mu : \tilde{X} \to \mathbb{R}$ the function defined by

$$\mathcal{D}_\mu(y) = \int_{\partial\tilde{X}} e^{B(y,\theta)} d\mu(\theta) \tag{4.1}$$

A computation shows that

$$Dd\mathcal{D}_\mu(y) = \int_{\partial\tilde{X}} (DdB(y, \theta) + DB(y, \theta) \otimes DB(y, \theta)) e^{B(y,\theta)} d\mu(\theta). \tag{4.2}$$

When $K_{\tilde{g}} \leq -1$ the Rauch comparison theorem says that for every $y \in \tilde{X}$, and $\theta \in \partial\tilde{X}$,

$$DdB(y,\theta) + DB(y,\theta) \otimes DB(y,\theta) \geq \tilde{g}. \tag{4.3}$$

We then get

$$Dd\mathcal{D}_\mu(y) \geq \mathcal{D}_\mu(y)\tilde{g}, \tag{4.4}$$

thus $Dd\mathcal{D}_\mu(y)$ is positive definite and $\mathcal{D}_\mu$ is strictly convex.

Lemma 4.1. *We have* $\lim_{y_k \to \partial\tilde{X}} \mathcal{D}_\mu(y) = +\infty$.

proof : Let $y_k \in \tilde{X}$ a sequence such that

$$\lim_{k\to\infty} y_k = \theta_0 \in \partial\tilde{X}. \tag{4.5}$$

As $supp(\mu) \cap (\partial\tilde{X} - \{\theta_0\}) \neq \emptyset$, there exists a compact subset $K \subset \partial\tilde{X} - \{\theta_0\}$ such that $\mu(K) > 0$ thus,

$$\int_{\partial\tilde{X}} e^{B(y_k,\theta)} d\mu \geq \int_K e^{B(y_k,\theta)} d\mu \to +\infty. \tag{4.6}$$

□

Corollary 4.2. *Let μ a finite borel measure on $\partial\tilde{X}$ whose support contains at least two points. The function $\mathcal{D}_\mu$ has a unique minimum. This minimum will be denoted by $\mathcal{C}(\mu)$.*

Let us now consider some discrete subgroup $C \subset Isom(\tilde{X}, \tilde{g})$. Recall that a family of Patterson measures $(\mu_x)_{x\in\tilde{X}}$ associated to C is a set of positive finite measures μ_x on $\partial\tilde{X}$, $x \in \tilde{X}$, such that the following holds for all $x \in \tilde{X}$, $\gamma \in C$,

$$\mu_{\gamma x} = \gamma_* \mu_x \tag{4.7}$$

$$\mu_x = e^{-\delta B(x,\theta)} \mu_o, \tag{4.8}$$

where $o \in \tilde{X}$ is a fixed origin, B the Busemann function associated to o and δ the critical exponent of C.

We assume now that $supp(\mu_o)$ contains at least two points and define the map $\tilde{F} : \tilde{X} \to \tilde{X}$ for $x \in \tilde{X}$ by

$$\tilde{F}(x) = \mathcal{C}(e^{-B(x,\theta)} \mu_x). \tag{4.9}$$

Here are a few notations. For a subspace E of $T_x\tilde{X}$, we will write $Jac_E\tilde{F}(x)$ the determinant of the matrix of the restriction of $D\tilde{F}(x)$ to E with respect to orthonormal bases of E and $D\tilde{F}(x)E$. For an integer p, we denote by $Jac_p\tilde{F}(x)$ the supremum of $|Jac_E\tilde{F}(x)|$ as E runs through the set of p-dimensional subspaces of $T_x\tilde{X}$.

Lemma 4.3. *The map $\tilde{F}$ is smooth, homotopic to the Identity and verifies for all $x \in \tilde{X}$, $\gamma \in C$ and $p \in [2, n = dim(X)]$,*

(i) $\tilde{F}(\gamma x) = \gamma \tilde{F}(x)$

(ii) $|Jac_p\tilde{F}(x)| \leq \left(\frac{(\delta+1)}{p}\right)^p$.

Proof :
The map

$$(x, y) \to \int_{\partial\tilde{X}} e^{B(y,\theta)-B(x,\theta)} d\mu_x(\theta) = \int_{\partial\tilde{X}} e^{B(y,\theta)-(\delta+1)B(x,\theta)} d\mu_o(\theta)$$

is smooth because $y \to B(y, \theta)$ is smooth.

For all x the map $y \to \int_{\partial\tilde{X}} e^{B(y,\theta)-(\delta+1)B(x,\theta)} d\mu_o(\theta)$ is strictly convex by (4.4) and tends to infinity when y tend to $\partial\tilde{X}$ (cf. lemma 4.1), thus the unique minimum $\tilde{F}(x)$ is a smooth function. The equivariance of $\tilde{F}$ comes from the cocycle relation $B(\gamma y, \gamma\theta) - B(\gamma x, \gamma\theta) = B(y, \theta) - B(x, \theta)$.

For each $x \in \tilde{X}$ let c_x be the geodesic in $\tilde{X}$ such that $c_x(0) = x$, $c_x(1) = \tilde{F}(x)$ and which is parametrized with constant speed. The map $\tilde{F}_t : \tilde{X} \to \tilde{X}$ defined by $\tilde{F}_t(x) = c_x(t)$ is a C-equivariant homotopy between $Id_{\tilde{X}}$ and $\tilde{F}$.

It remains to prove (ii).

The point $\tilde{F}(x)$ is characterized by

$$\int_{\partial\tilde{X}} DB(\tilde{F}(x), \theta) e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) = 0. \tag{4.10}$$

In order to simplify the notations we will write $B_{(x,\theta)}$ instead of $B(x, \theta)$ and we will denote ν_x the measure $e^{B(\tilde{F}(x),\theta)-B(x,\theta)}\mu_x$. We will also write $D\tilde{F}(u)$ instead of $D\tilde{F}(x)(u)$.

The differential of $\tilde{F}$ is characterized by the following: for $u \in T_x\tilde{X}$ and $v \in T_{\tilde{F}(x)}\tilde{X}$, one has

$$\int_{\partial\tilde{X}} [DdB_{(\tilde{F}(x),\theta)}(D\tilde{F}(u), v) + DB_{(\tilde{F}(x),\theta)}(v) DB_{(\tilde{F}(x),\theta)}(D\tilde{F}(u))] d\nu_x(\theta)$$

$$= (\delta + 1) \int_{\partial\tilde{X}} DB_{(\tilde{F}(x),\theta)}(v) DB_{(x,\theta)}(u) d\nu_x(\theta). \tag{4.11}$$

We define the quadratic forms k and h for $v \in T_{\tilde{F}(x)}\tilde{X}$ by

$$k(v, v) = \int_{\partial\tilde{X}} [DdB_{(\tilde{F}(x),\theta)}(v, v) + (DB_{(\tilde{F}(x),\theta)}(v))^2] d\nu_x(\theta). \tag{4.12}$$

and

$$h(v,v)=\int_{\partial\tilde{X}} DB_{(\tilde{F}(x),\theta)}(v)^2 d\nu_x(\theta). \tag{4.13}$$

The relation (4.11) writes, for $u\in T_x\tilde{X}$ and $v\in T_{\tilde{F}(x)}\tilde{X}$:

$$k(D\tilde{F}(u),v)=(\delta+1)\int_{\partial\tilde{X}} DB_{(\tilde{F}(x),\theta)}(v)DB_{(x,\theta)}(u)d\nu_x(\theta). \tag{4.14}$$

We defines the quadratic form h' on $T_x\tilde{X}$ for $u\in T_x\tilde{X}$ by

$$h'(u,u)=\int_{\partial\tilde{X}} DB_{(x,\theta)}(u)^2 d\nu_x(\theta), \tag{4.15}$$

and one derives from (4.14)

$$|k(D\tilde{F}(x)(u),v)|\le(\delta+1)h(v,v)^{1/2}h'(u,u)^{1/2}. \tag{4.16}$$

One now can estimate $Jac_p\tilde{F}(x)$. Let $P\subset T_x\tilde{X}$, $dimP=p$. If $D\tilde{F}(P)$ has dimension lower than p, then there is nothing to be proven. Let us assume that $dimD\tilde{F}(P)=p$. Denote by the same letters H' [resp. H and K] the selfadjoint operators (with respect to $\tilde{g}$) associated to the quadratic forms h' [resp. h, k] restricted to P [resp. $D\tilde{F}(P)$].

Let $(v_i)_{i=1}^p$ an orthonormal basis of $D\tilde{F}(P)$ which diagonalizes H and $(u_i)_{i=1}^p$ an orthonormal basis of P such that the matrix of $K\circ D\tilde{F}(x):P\to D\tilde{F}(P)$ is triangular. Then,

$$detK.|Jac_P\tilde{F}(x)|\le(\delta+1)^p(\Pi_{i=1}^p h(v_i,v_i)^{1/2})(\Pi_{i=1}^p h'(u_i,u_i)^{1/2}) \tag{4.17}$$

thus,

$$detK.|Jac_P\tilde{F}(x)|\le(\delta+1)^p\Big(\frac{TraceH}{p}\Big)^{p/2}\Big(\frac{TraceH'}{p}\Big)^{p/2}. \tag{4.18}$$

In these inequalities one can normalize the measures

$$\nu_x=e^{B(\tilde{F}(x),\theta)-B(x,\theta)}\mu_x$$

such that their total mass equals one, which gives

$$traceH=\Sigma_{i=1}^p h(v_i,v_i)\le 1, \tag{4.19}$$

the last inequality coming from the fact that for all $\theta\in\partial\tilde{X}$,

$$\Sigma_{i=1}^p DB(\tilde{F}(x),\theta)(v_i)^2\le||\nabla B(\tilde{F}(x),\theta)||^2=1 \tag{4.20}$$

and from the previous normalization.

Similarly,

$$traceH' = \Sigma_{i=1}^{p} h'(u_i, u_i) \leq 1. \tag{4.21}$$

We then obtain with (4.18)

$$detK.|Jac_P\tilde{F}(x)| \leq \Big(\frac{\delta+1}{p}\Big)^p. \tag{4.22}$$

Thanks to (4.3), we have $detK \geq 1$, so

$$|Jac_P\tilde{F}(x)| \leq \Big(\frac{\delta+1}{p}\Big)^p. \tag{4.23}$$

We get (ii) by taking the supremum in P. □

As the map $\tilde{F} : \tilde{X} \to \tilde{X}$ is C-equivariant, then for every subgroup $C' \subset C$, $\tilde{F}$ gives rise to a map $F' : \tilde{X}/C' \to \tilde{X}/C'$ and so does the homotopy $\tilde{F}_t$ between $\tilde{F}$ and $Id_{\tilde{X}}$.

Corollary 4.4. *The map $F' : \tilde{X}/C' \to \tilde{X}/C'$ is homotopic to the Identity map and verifies for all $x \in \tilde{X}/C'$ and $p \in [2, n = dimX]$*

$$|JacF'_p(x)| \leq \Big(\frac{\delta+1}{p}\Big)^p.$$

Let $C \subset Isom(\tilde{X}, \tilde{g})$ as above, ie such that the support of the Patterson-Sullivan measures contains at least two points and with critical exponent δ. Let $C' \subset C$ be a subgroup.

Let us consider an compact hypersurface $Z' \subset \tilde{X}/C'$.

Denote $F'^k = F' \circ F' \circ ..... \circ F'$ the composition of F' k-times.

Let us write $g_k = (F'^k)^*g$, where g is the metric on $\tilde{X}/C'$ induced by $\tilde{g}$. The symmetric 2-tensor g_k may not be a riemanniann metric on $\tilde{X}/C'$ nor its resriction to Z', so we have to modify it. For $\epsilon > 0$, the following symmetric 2-tensor $g_{\epsilon,k}$ is a riemanniann metric on $\tilde{X}/C'$ and so is its restriction $h_{\epsilon,k}$ to Z'.

$$g_{\epsilon,k} = g_k + \epsilon^2 g. \tag{4.24}$$

Lemma 4.5. *Let $h_{\epsilon,k}$ be the restriction of $g_{\epsilon,k}$ to the hypersurface Z' and $g_{Z'}$ the restriction of g to Z'. Let $\Phi_{\epsilon,k} : Z' \to \mathbb{R}$ the density defined for all $x \in Z'$ by $dv_{h_{\epsilon,k}}(x) = \Phi_{\epsilon,k}(x) dv_{g_{Z'}}(x)$. For any sequence ϵ_k such that* $\lim_{k\to\infty} \epsilon_k = 0$, *there exists a sequence ϵ'_k,* $\lim_{k\to\infty} \epsilon'_k = 0$, *such that for all $x \in Z'$,*

$$0 < \Phi_{\epsilon'_k,k}(x) \leq |Jac_{n-1}F'^k(x)| + \epsilon_k.$$

In particular,

$$\Phi_{\epsilon'_k,k}(x) \leq \Big(\frac{\delta+1}{n-1}\Big)^{k(n-1)} + \epsilon_k$$

and

$$vol(Z', h_{\epsilon'_k,k}) \leq \Big[\Big(\frac{\delta+1}{n-1}\Big)^{k(n-1)} + \epsilon_k\Big] vol(Z', g_{Z'}).$$

Corollary 4.6. *Under the above asumptions, if $\delta < n-2$ there exists a sequence ϵ'_k such that $\lim_{k\to\infty}\epsilon'_k = 0$, and $lim_{k\to\infty} vol(Z', h_{\epsilon'_k,k}) = 0$.*

Proof of lemma 4.5 :

Let us fix k an integer. Let $x \in Z'$ and $u \in T_xZ'$. We have $g_{\epsilon,k}(u,u) = h_{\epsilon,k}(u,u) = g(DF'^k(x)(u), DF'^k(x)(u)) + \epsilon^2 g(u,u)$ thus $h_{\epsilon,k}(u,u) = g(A_{x,\epsilon}u, u)$ where $A_{x,\epsilon} \in End(T_xZ')$ is the self adjoint operator $A_x = DF'^k(x)^* \circ DF'^k(x) + \epsilon^2 Id$, with $DF'^k(x)^*$ the adjoint of $DF'^k(x) : (T_xZ', g(x)) \to (DF'^k(x)(T_xZ'), g(F'^k(x)))$.

By compactness of Z' and continuity of $A_{x,\epsilon}$, there exist ϵ'_k such that

$$\Phi_{\epsilon'_k,k}(x) = detA^{1/2}_{x,\epsilon'_k} \leq detA_{x,0} + \epsilon_k,$$

thus

$$\Phi_{\epsilon'_k,k}(x) \leq |Jac_{n-1}F'^k(x)| + \epsilon_k.$$

The lemma then follows from Corollary 4.4. □

5. Proof of Theorem 1.2

This section is devoted to the proof of the Theorem 1.2. Let Γ be a discrete cocompact group of isometries of a n-dimensional Cartan-Hadamard manifold $(\tilde{X}, \tilde{g})$ whose sectional curvature satisfies $K_{\tilde{g}} \leq -1$. We assume that $\Gamma = A *_C B$. At the end of section 2 we constructed a subgroup $C' \subset C$ and an orientable hypersurface $\tilde{Z}' \subset \tilde{X}$ such that $C'.\tilde{Z}' = \tilde{Z}'$ and $Z' = \tilde{Z}'/C'$ is compact in $\tilde{X}/C'$. Moreover Z' is essential in $\tilde{X}/C'$ ie $i_*([Z']) \neq 0$ where $i_* : H_{n-1}(Z', \mathbb{R}) \to H_{n-1}(\tilde{X}/C', \mathbb{R})$ is the morphism induced on homology groups by the inclusion $i : Z' \to \tilde{X}/C'$ and $[Z']$ the fundamental class of Z'.

5.1 Proof of the inequality

We now prove the inequality in the theorem 1.2. Let us assume that $\delta < n-2$ and derive a contradiction. Let $h_{\epsilon'_k,k}$ the sequence of metric defined on Z' in lemma 4.5, then by corollary 4.6 we have

$$\lim_{k\to\infty} vol(Z', h_{\epsilon'_k,k}) = 0. \tag{5.1}$$

We now show that the systole of the metric $h_{\epsilon'_k,k}$ on Z' is bounded below independently of k. Recall that the systole of $i : Z' \to \tilde{X}/C'$ at a point $z \in Z'$ with respect to a metric $h_{\epsilon,k}$ can be defined by

$$\mathrm{sys_i(Z', h_{\epsilon,k}, z) = inf_{\gamma\in C'} dist_{(\tilde{Z}', \tilde{h}_{\epsilon,k})}(\tilde{z}, \gamma\tilde{z})} \tag{5.2}$$

where $\tilde{z}$ is any lift of z and $\tilde{h}_{\epsilon,k}$ the lift on $\tilde{Z}'$ of $h_{\epsilon,k}$, (cf. lemma 3.4).

Let $\alpha(t)$ be a minimizing geodesic between $\tilde{z}$ and $\gamma\tilde{z}$ on $(Z', \tilde{h}_{\epsilon,k})$. By definition of $\tilde{h}_{\epsilon,k}$ we have

$$\mathrm{dist}_{(\tilde{Z}', \tilde{h}_{\epsilon,k})}(\tilde{z}, \gamma\tilde{z}) \geq l_{\tilde{g}}(\tilde{F}^k \circ \alpha) \tag{5.3}$$

where $l_{\tilde{g}}$ stands for the lengh with respect to $\tilde{g}$ on $\tilde{X}$.
We get

$$\mathrm{dist}_{(\tilde{Z}', \tilde{h}_{\epsilon,k})}(\tilde{z}, \gamma\tilde{z}) \geq \mathrm{dist}_{(\tilde{X}, \tilde{g})}(\tilde{F}^k(\tilde{z}), \gamma\tilde{F}^k(\tilde{z})) \geq \rho \tag{5.4}$$

where ρ is the injectivity radius of $\tilde{X}/C'$.
We then have

$$\mathrm{sys}_i(Z', h_{\epsilon,k}) \geq \rho, \tag{5.5}$$

and thanks to the Theorem 3.3 we obtain

$$\mathrm{vol}(Z', h_{\epsilon'_k,k}) \geq C_n \rho^{n-1} \tag{5.6}$$

which contradicts (5.1).
□

5.2 Proof of the equality case

There will be several steps.

Step 1: The limit set of C is contained in a topological equator.

Step 2: The weak tangent to $\partial\tilde{X}$ and $\Lambda_{C'}$.

Step 3: The limit set $\Lambda_{C'}$ of C' and the limit set Λ_C of C are equal to a topological equator.

Step 4: C' and C are convex cocompact.

Step 5: C preserves a copy of the real hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}}$ totally geodesically embedded in $\tilde{X}$.

Step 6: Conclusion

Step 1: The limit set Λ_C of C is contained in a topological equator.

Let $x \in \tilde{X}$ and $E \subset T_x\tilde{X}$ a codimension one subspace. For each $u \in T_x\tilde{X}$, $\tilde{g}(u, u) = 1$, one considers the geodesic c_u defined by $c_u(0) = x$ and $\dot{c_u}(0) = u$. We define the equator $E(\infty)$ associated to E as the subset of $\partial\tilde{X}$

$$E(\infty) = \{c_u(+\infty)/u \in E\} \tag{5.7}$$

Our goal is to prove the existence of a point $x \in \tilde{X}$ and an hyperplane $E \subset T_x\tilde{X}$ such that the limit set Λ_C satisfies $\Lambda_C \subset E(\infty)$.

Recall that $C' \subset C$ globally preserves an hypersurface $\tilde{Z}'$ such that $\tilde{Z}'/C' \subset \tilde{X}/C'$ is compact and essential.

Let us also recall that we have constructed a C-equivariant map $\tilde{F} : \tilde{X} \to \tilde{X}$ such that, for all $x \in \tilde{X}$,

$$|Jac_{n-1}\tilde{F}(x)| \leq \Big(\frac{\delta+1}{n-1}\Big)^{n-1} \tag{5.8}$$

where the critical exponent δ of C satisfies $\delta = n-2$, thus

$$|Jac_{n-1}\tilde{F}(x)| \leq 1. \tag{5.9}$$

The step 1 follows from the two following Propositions.

Proposition 5.1. *Let $x \in \tilde{X}$ such that $|Jac_{n-1}\tilde{F}(x)| = 1$. Then there exists $E \subset T_x\tilde{X}$ such that the limit set Λ_C satisfies $\Lambda_C \subset E(\infty)$. Moreover, $\tilde{F}(x) = x$ and $D\tilde{F}(x)$ is the orthogonal projector onto E.*

Proposition 5.2. *There exists $x \in \tilde{X}$ such that $|Jac_{n-1}\tilde{F}(x)| = 1$.*

Proof of Proposition 5.1

Let $x \in \tilde{X}$ such that $|Jac_{n-1}\tilde{F}(x)| = 1$. By definition we have a subspace $E \subset T_x\tilde{X}$ such that $|Jac_E\tilde{F}(x)| = 1$. By (4.18) and detK ≥ 1 we have,

$$|Jac_E\tilde{F}(x)| \leq (n-1)^{n-1}\Big(\frac{traceH}{n-1}\Big)^{\frac{n-1}{2}}\Big(\frac{traceH'}{n-1}\Big)^{\frac{n-1}{2}}$$

$$\leq (n-1)^{n-1}(\frac{1}{n-1})^{n-1}. \tag{5.10}$$

In particular as $|Jac_E\tilde{F}(x)| = 1$, we have equality in the inequalities (5.10), thus, $traceH = trace(h) = 1$, and

$$H = \frac{1}{n-1}Id_{D\tilde{F}(x)(E)}. \tag{5.11}$$

Let us recall that the quadratic form h is defined by

$$h(v,v) = \int_{\partial\tilde{X}} DB(\tilde{F}(x),\theta)(v)^2 e^{B(\tilde{F}(x),\theta)-B(x,\theta)}d\mu_x(\theta)$$

where μ_x is the Patterson-Sullivan measure of C normalized by

$$\int_{\partial\tilde{X}} e^{B(\tilde{F}(x),\theta)-B(x,\theta)}d\mu_x(\theta) = 1. \tag{5.12}$$

We then have

$$1 = trace(h) = traceH = \Sigma_{i=1}^{n-1} h(v_i, v_i) =$$

$$= \int_{\partial \tilde{X}} \Sigma_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta)$$

$$\leq \int_{\partial \tilde{X}} ||DB(\tilde{F}(x), \theta)||^2 e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) \leq 1,$$

because $\Sigma_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 \leq ||DB(\tilde{F}(x), \theta)||^2 = 1$
for all $\theta \in \partial \tilde{X}$.

Therefore for μ_x-almost all $\theta \in supp(e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta)) = supp(\mu_x)$, we have

$$\Sigma_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 = ||DB(\tilde{F}(x), \theta)||^2 = 1. \tag{5.13}$$

In (5.12), $\Sigma_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2$ represents the square of the norm of the projection of $\nabla B(\tilde{F}(x), \theta)$ on E.

By continuity of $B(x, \theta)$ in θ one then gets $\Lambda_C = supp(\mu_x) \subset E(\infty)$.

Let us now prove that $\tilde{F}(x) = x$. When $Jac\tilde{F}_E(x) = 1$, we have equality in the Cauchy-Schwarz inequality (4.16), therefore for each $i = 1, ...n-1$ and $\theta \in \Lambda_C$ we get $DB(\tilde{F}(x), \theta)(v_i) = DB(x, \theta)(u_i)$. Therefore we deduces from (5.13) that $\nabla B(x, \theta) = \Sigma_{i=1}^{i=n-1} DB(x, \theta)(u_i) u_i$, which imply with (4.10) that

$$\int_{\partial \tilde{X}} DB(x, \theta) e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) = 0. \tag{5.14}$$

On the other hand, as $\int_{\partial \tilde{X}} DB(\tilde{F}(x), \theta) e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) = 0$ and $H = \frac{1}{n-1} Id_{D\tilde{F}(x)(E)}$, the support of μ_x cannot be just a pair of points, therefore the barycenter of the measure $e^{B(\tilde{F}(x),\theta)-B(x,\theta)} \mu_x$ defined in [2] is well defined and characterized as the point $z \in \tilde{X}$ such that

$$\int_{\partial \tilde{X}} DB(z, \theta) e^{B(\tilde{F}(x),\theta)-B(x,\theta)} d\mu_x(\theta) = 0,$$

thus (5.14) and (4.10) imply $x = \tilde{F}(x)$. $\square$

Proof of Proposition 5.2 :

If we knew that there exists a minimizing hypersurface Z_0 in the homology class of Z, then every points $x \in Z_0$ would verify $|Jac_{n-1}\tilde{F}(x)| = 1$. We unfortunately don't know if there exists such a minimizing hypersurface nor a minimizing current in the homology class of Z. Instead we will consider an $L^2(\tilde{X}/C')$ harmonic $(n-1)$-form dual to the homology class of Z.

We need the following lemmas in order to prove the existence of such a dual form.

Let $\lambda_1(\tilde{X}/C')$ be the bottom of the spectrum of the Laplacian on $(\tilde{X}, \tilde{g})$, ie.

$$\lambda_1(\tilde{X}/C') = inf_{u \in C_0^\infty(\tilde{X}/C')} \{ \frac{\int_{\tilde{X}/C'} |du|^2}{\int_{\tilde{X}/C'} u^2} \}. \tag{5.15}$$

Lemma 5.3. *Let $C \subset Isom(\tilde{X}, \tilde{g})$ a discrete group of isometries with critical exponent $\delta = n-2$ where $(\tilde{X}, \tilde{g})$ is an n-dimensional Cartan-Hadamard manifold of sectional curvature $K_{\tilde{g}} \leq -1$. Then for any subgroup $C' \subset C$ we have $\lambda_1(\tilde{X}/C') \geq n-2$.*

Proof :

Thanks to a theorem of Barta, cf.[17] Theorem 2.1, the lemma boils down to finding a positive function $c : \tilde{X}/C' \to \mathbb{R}_+$ such that $\Delta c(x) \geq (n-2)c(x)$. Here, the laplacian Δ is the positive operator ie. $\Delta c = -traceDdc$. We consider the smooth function $\tilde{c} : \tilde{X} \to \mathbb{R}_+$ defined by $\tilde{c}(x) = \mu_x(\partial \tilde{X})$ where $\{\mu_x\}_{x \in \tilde{X}}$ is a family of Patterson-Sullivan measure of C. The function $\tilde{c}$ is C-equivariant therefore it defines a map $c : \tilde{X}/C' \to \mathbb{R}_+$ for any subgroup $C' \subset C$. Let us show

$$\Delta c(x) \geq \delta(n-1-\delta)c(x) = n-2. \tag{5.16}$$

We have

$$\tilde{c}(x) = \int_{\partial \tilde{X}} e^{-\delta B(x,\theta)} d\mu_o(\theta)$$

therefore

$$\Delta \tilde{c}(x) = \int_{\partial \tilde{X}} [-\delta \Delta B(x,\theta) - \delta^2] d\mu_x(\theta).$$

The sectional curvature $K_{\tilde{g}}$ of $(\tilde{X}, \tilde{g})$ satisfies $K_{\tilde{g}} \leq -1$ we thus have $-\Delta B(x,\theta) \geq n-1$ and as $\delta = n-2$ we get

$$\Delta \tilde{c}(x) \geq [\delta(n-1) - \delta^2]\tilde{c}(x) = (n-2)\tilde{c}(x).$$

□

The following lemma is due to G.Carron and E.Pedon, [8]. For a complete riemannian manifold Y, we denote $H_c^1(Y, \mathbb{R})$ the first cohomology group generated by diferential forms with compact support.

Lemma 5.4 ([8], Lemme 5.1)**.** *Let Y be a complete riemannian manifold all ends of whose having infinite volume and such that $\lambda_1(Y) > 0$, then the natural morphism*

$$H_c^1(Y, \mathbb{R}) \to H_{L^2}^1(Y, \mathbb{R})$$

is injective. In particular any $\alpha \in H_c^1(Y, \mathbb{R})$ admits a representative $\bar{\alpha}$ which is in $L^2(Y, \mathbb{R})$.

Corollary 5.5. *Let C' be as above and assume that there exists a compact essential hypersurface $Z' \subset \tilde{X}/C'$. Then there exists an harmonic $n-1$-form ω in $L^2(\tilde{X}/C')$ such that $\int_{Z'} \omega \neq 0$.*

Proof :

Let $\alpha \in H^1_c(\tilde{X}/C', \mathbb{R})$ a Poincaré dual of $[Z'] \in H_{n-1}(\tilde{X}/C', \mathbb{R})$. By definition of α, for any $\beta \in H^{n-1}(\tilde{X}/C', \mathbb{R})$, one has

$$\int_{Z'} \beta = \int_{\tilde{X}/C'} \beta \wedge \alpha, \tag{5.17}$$

([5] p.51, note that $\tilde{X}/C'$ has a " finite good cover").

After Lemma 5.4 , α admits a non trivial harmonic representative $\bar{\alpha}$ in $L^2(\tilde{X}/C')$. (In order to apply the Lemma 5.4, one has to check that all ends of $\tilde{X}/C'$ have infinite volume, ie for a compact $K \subset \tilde{X}/C'$ each unbounded connected component of $\tilde{X}/C' - K$ has infinite volume: this comes from the fact that the injectivity radius of $\tilde{X}/C'$ is bounded below by the injectivity radius of $X = \tilde{X}/\Gamma$ and the sectional curvature bounded above by -1.) The $(n-1)$-harmonic form $\omega = *\bar{\alpha}$, where $*$ is the Hodge operator, is in $L^2(\tilde{X}/C')$ and verifies after (5.17)

$$\int_{Z'} \omega = \int_{\tilde{X}/C'} \omega \wedge \bar{\alpha} = \int_{\tilde{X}/C'} \omega \wedge *\omega = ||\omega||^2_{L^2(\tilde{X}/C')} \neq 0. \tag{5.18}$$

□

We can now prove the proposition 5.2. Let us briefly describe the idea. We consider the iterates F'^k of $F' : \tilde{X}/C' \to \tilde{X}/C'$. As F' is homotopic to the identity map, $F'^k(Z')$ is homologous to Z' and if ω is the harmonic form of the corollary 5.5 we have

$$\int_{Z'} (F'^k)^*(\omega) = \int_{Z'} \omega = a \neq 0. \tag{5.19}$$

We don't know if $F'^k(Z')$ converges or stays in a compact subset of $\tilde{X}/C'$ but we will show that $F'^k(Z')$ cannot entirely diverge in $\tilde{X}/C'$ and that there exists a $z' \in Z'$ such that $F'^k(z')$ subconverges to a point $x \in \tilde{X}/C'$ with $|Jac_{n-1}F'(x)| = 1$.

¿From (5.19) one gets

$$0 < |a| = |\int_{Z'} (F'^k)^*(\omega)| \leq \int_{Z'} |Jac_{TZ'} F'^k(z)|.||\omega(F'^k(z)||dz \tag{5.20}$$

where

$$|Jac_{TZ'} F'^k(z)| = ||DF'^k(z)(u_1) \wedge DF'^k(z)(u_2) \wedge ... \wedge DF'^k(z)(u_{n-1})||$$

and $(u_1, .., u_{n-1})$ is an orthonormal basis of $T_z(Z')$.

Let us define $\mathcal{B} = \{z \in Z', |JacF'^k_{TZ'}(z)|$ *does not converge to 0* $\}$.

For $z \in Z'$ we define the sequence z_k by $z_0 = z$ and $z_k = F'(z_{k-1}) = F'^k(z) \in \tilde{X}/C'$.

Lemma 5.6. *There exists $z \in \mathcal{B}$ and a subsequence z_{k_j} such that z_{k_j} converges to a point $x \in \tilde{X}/C'$ with $|Jac_{n-1}(x)| = 1$.*

Proof :

We first remark that $\lim_{x\to\infty} ||\omega(x)|| = 0$. This follows the following facts: ω is harmonic, $\omega \in L^2(\tilde{X}/C')$ and the injectivity radius of $\tilde{X}/C'$ is bounded below by a positive constant.

Let us assume that for all $z \in \mathcal{B}$ the sequence z_k diverges in $\tilde{X}/C'$. Then we have, for all $z \in \mathcal{B}$,

$$||\omega(z_k)|| = ||\omega(F'^k(z))|| \to 0 \tag{5.21}$$

whenever k tends to ∞ because of the previous remark.

On the other hand, as $||\omega(F'^k(z))|| \leq C$ and $|Jac_{n-1}F'^k| \leq 1$, it follows from (5.20)

$$\lim_{k\to\infty} |\int_{Z'} (F'^k)^*(\omega)| \leq$$

$$\lim_{k\to\infty} [\int_{\mathcal{B}} ||\omega(F'^k(z))||dz + C\int_{Z'-\mathcal{B}} |Jac_{Z'}F'^k(z)|dz = 0$$

which contradicts our assumption.

Thus there exists a point $z \in Z'$ such that

$$|Jac_{Z'}F'^k(z)| \to \alpha \neq 0 \tag{5.22}$$

and such that there exists a subsequence $z_{k_j} = F'^{k_j}(z)$ with

$$lim_{j\to\infty} z_{k_j} = x \in \tilde{X}/C'. \tag{5.23}$$

The property (5.22) comes from the fact that the sequence $|Jac_{Z'}F'^k(z)|$ doesn't tend to zero and is decreasing (because $|Jac_{n-1}F'| \leq 1$).

Let us define

$$E_0 = T_z Z', E_1 = DF'(z)(E_0)$$

and

$$E_k = DF'(z_{k-1})(E_{k-1}) \subset T_{z_k}(\tilde{X}/C').$$

As $z_{k_j} \to x$ we can assume, after extracting again a subsequence, that $E_{k_j} \to E \subset T_x(\tilde{X}/C')$. On the other hand we also have

$$|Jac_{Z'}F'^k(z)| = |Jac_{E_{k-1}}F'(z_{k-1})||Jac_{E_{k-2}}F'(z_{k-2})|...|Jac_{E_0}F'(z)| \tag{5.24}$$

We know that $|Jac_{E_k} F'(z_k)| = 1 - \epsilon_k$ where $0 \leq \epsilon_k < 1$. As $z \in \mathcal{B}$, we have

$$\lim_{k \to \infty} \pi_{j=1}^{k} (1 - \epsilon_j) = \alpha > 0$$

therefore $\lim_{k\to\infty} \epsilon_k = 0$ and by continuity we have $|Jac_E F'(x)| = 1$. $\square$

Now we can finish the proof of the step 1. Let consider a lift of x in $\tilde{X}$ and E in $T\tilde{X}$ that we again call x and E. Then we have $|Jac_E \tilde{F}'(x)| = 1$. $\square$

Let us remark that corollary 4.4 and (5.19) give another proof of the inequality $\delta \geq n-2$, which does not use the isosystolic inequality, ie. Theorem 3.3.

Step 2 : The weak tangent of $\partial\tilde{X}$ and Λ_C

We first recall the definition of the Gromov-distance on $\partial\tilde{X}$.

For two arbitrary points θ and θ' in $\partial\tilde{X}$ let us define

$$l(\theta, \theta') = inf\{t > 0 / dist(\alpha_\theta(t), \alpha_{\theta'}(t)) = 1\} \tag{5.25}$$

and

$$d(\theta, \theta') = e^{-l(\theta,\theta')} \tag{5.26}$$

then d is a distance on $\partial\tilde{X}$.

We now recall a few definitions following [4]. A complete metric space $(S, \bar{d})$ is a weak tangent of a metric space (Z, d) if there exist a point $0 \in S$, a sequence of points $z_k \in Z$ and a sequence of positive real numbers $\lambda_k \to \infty$ such that the sequence of pointed metric spaces $(Z, \lambda_k d, z_k)$ converges in the pointed Gromov-Hausdorff topology to $(S, \bar{d}, 0)$ where $(Z, \lambda_k d)$ stands for the set Z endowed with the rescaled metric $\lambda_k d$.

Let us recall that the sequence of metric spaces (Z_k, d_k, z_k) converges to $(S, \bar{d}, 0)$ in the pointed Gromov-Hausdorff topology if the following conditions hold, (cf. (B-B-I), definition 8.1.1).

Definition 5.7. *We say that the sequence of metric spaces (Z_k, d_k, z_k) converges to $(S, \bar{d}, o)$ if for any $R > 0$, $\epsilon > 0$ there exists k_0 such that for any $k \geq k_0$ there exists a (non necessary continuous) map $f : B(z_k, R) \to S$ such that*
(i) $f(z_k) = 0$,
for any two points x and y in $B(z_k, R)$,
(ii) $|\bar{d}(f(x), f(y)) - d_k(x, y)| \leq \epsilon$,
and
(iii) the ϵ-neighborhood of the set $f(B(z_k, R)$ contains $B(0, R - \epsilon)$.

In the previous definition, $B(z_k, R)$ stands for the ball of radius R centered at the point z_k in (Z_k, d_k).

For a metric space (Z, d) we will denote $WT(Z, d)$ the set of weak tangents of (Z, d).

Let Γ a cocompact group of isometries of $(\tilde{X}, \tilde{g})$ a n-dimensional Cartan Hadamard manifold of sectional curvature $K_{\tilde{g}} \leq -1$ and C' a subgroup of Γ. The limit set Λ_Γ of Γ is the full boundary $\partial\tilde{X}$ namely a topological $(n-1)-$dimensional sphere S^{n-1}. We endow $\partial\tilde{X}$ with the Gromov distance d defined in (5.26). In [4] Lemma 5.2, M.Bonk and B.Kleiner show among other properties the following

Lemma 5.8. *For any weak tangent space $(S, \bar{d})$ in $WT((\partial\tilde{X}, d))$, S is homeomorphic to $\partial\tilde{X}$ less a point, thus to $\mathbb{R}^{n-1}$.*

In fact the crucial asumption in the above lemma, coming from the cocompacness of Γ, is the property that any triple of points in $\partial\tilde{X}$ can be uniformly separated by an element of Γ, ie. there is $\delta > 0$ such that for any three points $\theta_1, \theta_2, \theta_3 \in \partial\tilde{X}$ there exists a $\gamma \in \Gamma$ such that $d(\gamma\theta_i, \gamma\theta_j) \geq \delta$ for all $1 \leq i \neq j \leq 3$. Following the argument of M.Bonk and B.Kleiner one can show that if C' is a subgroup of Γ such that one weak tangent $(S, \bar{d})$ of $(\Lambda_{C'}, d)$ is in $WT(\partial\tilde{X}, d)$ and enough triples of points of $\Lambda_{C'}$ can be uniformly separated by elements of C', then S is homeomorphic to $\Lambda_{C'}$ less a point. In particular, $\Lambda_{C'}$ is homeomorphic to $\partial\tilde{X}$.

Lemma 5.9. *Let $\mathcal{L} \subset \partial\tilde{X}$ be a closed C'-invariant set and $\theta_0 \in \mathcal{L}$. We assume that there exist a sequence of positive real numbers $\lambda_k \to \infty$ such that the sequence of pointed metric spaces $(\mathcal{L}, \lambda_k d, \theta_0)$ converges in the pointed Gromov-Hausdorff topology to $(S, \bar{d}, 0)$ where $(S, \bar{d}, 0)$ is a weak tangent of $(\partial\tilde{X}, d)$. We also assume that there exist positive constants C and δ, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{L}$ and a sequence of elements $\gamma_k \in C'$ such that $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C\lambda_k^{-1}$ and $d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \delta$ for all $0 \leq i \neq j \leq 2$. Then, S is homeomorphic to $\mathcal{L}$ less a point. In particular $\mathcal{L}$ is homeomorphic to $\partial\tilde{X}$.*

The proof of this lemma is postponed in the Appendix.

Step 3 : The limit set $\Lambda_{C'}$ of C' and the limit set Λ_C of C are equal to a topological equator.

We have shown in step 1 that Λ_C is a subset of some topological equator $E(\infty)$.

Let $o \in \tilde{X}$ and $E \subset T_o\tilde{X}$ be such that $|Jac_E\tilde{F}(o)| = 1$ and $E(\infty)$ is the equator associated to E.

Recall that there exists a subgroup C' of C which globally preserves an hypersurface $\tilde{Z}' \subset \tilde{X}$ and that $\tilde{Z}'/C' \subset \tilde{X}/C'$ is compact. Furthermore $\tilde{Z}'$ separates $\tilde{X}$ into two connected components $\tilde{U}$ et $\tilde{V}$. We can assume that $\tilde{U}$ and $\tilde{V}$ are globally invariant by C' after having replaced C' by an index 2 subgroup.

The limit set $\Lambda_{C'}$ of C' is contained in Λ_C, therefore $\Lambda_{C'} \subset E(\infty)$.

We will show that $\Lambda_{C'} = E(\infty)$.

For any subset $W \in \tilde{X}$ we define the boundary at infinity ∂W of W by

$$\partial W = Cl(W) \cap \partial \tilde{X} \tag{5.27}$$

where $Cl(W)$ stands for the closure of W in $\tilde{X} \cup \partial \tilde{X}$.

As $\tilde{Z}'/C'$ is compact, $\tilde{Z}'$ is at bounded distance of the orbit $C'z$ of some point z in $\tilde{Z}'$, thus

$$Cl(\tilde{Z}') \cap \partial \tilde{X} = \Lambda_{C'} \tag{5.28}$$

By definition we have $\Lambda_{C'} \subset \partial \tilde{U}$ and $\Lambda_{C'} \subset \partial \tilde{V}$.

Lemma 5.10. *Let us assume that $\Lambda_{C'} \neq E(\infty)$, then either $\partial \tilde{U} = \Lambda_{C'}$ or $\partial \tilde{V} = \Lambda_{C'}$.*

Proof: We know that $\Lambda_{C'} \subset \partial \tilde{U}$ and $\Lambda_{C'} \subset \partial \tilde{V}$.

Let us assume that the conclusion of the lemma is not true, so there is $\zeta \in \partial \tilde{U} - \Lambda_{C'}$ and $\theta \in \partial \tilde{V} - \Lambda_{C'}$.

As $\Lambda_{C'} \subset E(\infty)$, $\Lambda_{C'} \neq E(\infty)$ and $\partial \tilde{X}$ is a sphere, any two points of $\partial \tilde{X} - \Lambda_{C'}$ can be joined by a continuous path contained in $\partial \tilde{X} - \Lambda_{C'}$ and so does ζ and θ, joined by such a path α.

The set $Z' \cup \Lambda_{C'}$ is a closed subset of $\tilde{X} \cup \partial \tilde{X}$ thus there is an open connected neighborhood W of α in $\tilde{X} \cup \partial \tilde{X}$ contained in the complementary of $Z' \cup \Lambda_{C'}$.

As ζ and θ can be approximated by points in $\tilde{U}$ and $\tilde{V}$ respectively there exist points $x \in \tilde{U} \cap W$ and $y \in \tilde{V} \cap W$ that can be joined by a continuous path by connectedness of W, which leads to a contradiction. $\square$

Remark 5.11. *In fact, we are going to show that under the assumption $\delta(C') = n - 2$, it is impossible to have $\partial \tilde{U} = \Lambda_{C'}$ or $\partial \tilde{V} = \Lambda_{C'}$.*

For any $x \in \tilde{X}$ and $\theta \in \partial \tilde{X}$ let us denote $HB(x, \theta)$ the open horoball centered at θ and passing through x.

Lemma 5.12. *Let us assume that $\partial \tilde{U} = \Lambda_{C'}$. Then there exist $\theta_0 \in \Lambda_{C'}$ and $z' \in \tilde{Z}'$ such that $HB(z', \theta_0) \subset \tilde{U}$.*

Proof : Let us recall that $\tilde{X}/C' - Z' = U \cup V$ where $U = \pi(\tilde{U})$ $V = \pi(\tilde{V})$ and $\pi : \tilde{X} \to \tilde{X}/C'$ is the projection .

We know that U and V are unbounded. Let x_n a sequence of points in U such that $dist(x_n, Z') \to \infty$. Let $z_n \in Z'$ such that $dist(x_n, Z') = dist(x_n, z_n)$. We consider a fundamental domain $D \subset \tilde{Z}'$ of C'. There exist lifts $\tilde{z}_n \in D$ and $\tilde{x}_n \in \tilde{U}$ such that $dist(\tilde{x}_n, Z') = dist(\tilde{x}_n, \tilde{z}_n)$ tends to infinity.

By compactness we can assume that a subsequence $\tilde{x}_{n_j}$ converges to a point $\theta_0 \in \partial \tilde{X}$ and $\tilde{z}_{n_j}$ also converges to a point $\tilde{z} \in \bar{D}$. Furthermore the sequence of open balls $B(\tilde{x}_{n_j}, dist(\tilde{x}_{n_j}, \tilde{z}_{n_j})) \subset \tilde{U}$ converges to the open horoball $HB(\theta_0, \tilde{z}) \in \tilde{U}$. $\square$

Proposition 5.13. *$\Lambda_{C'} = E(\infty)$ and $\Lambda_C = E(\infty)$.*

We first describe the idea of the proof and next state some facts we will need in order to do it.

As $\Lambda_{C'} \subset \Lambda_C \subset E(\infty)$ the proposition boils down to proving that $\Lambda_{C'} = E(\infty)$. Let us assume $\Lambda_{C'} \neq E(\infty)$ and find a contradiction.

We will show that for any sequence θ_i converging to θ_0 the geodesic starting at a point o such that $|Jac_{n-1}\tilde{F}(o)| = 1$ and ending at θ_i crosses the hypersurface $\tilde{Z}'$ in a point z_i. For an **appropriate choice** of such a sequence θ_i (roughly speaking, the sequence θ_i is chosen to be converging to θ_0 "transversally to $\Lambda_{C'}$"), the shadow (defined below) projected from o through some geodesic ball $B(z_i, r)$ will not intersect $\Lambda_{C'}$. On the other hand this shadow has to meet the limit set $\Lambda_{C'}$ because of the shadow lemma of D. Sullivan, which leads to a contradiction.

Precisely, by Lemma (5.10) and Lemma (5.12) we know that $\partial\tilde{U} = \Lambda_{C'}$ and that there exists an open horoball $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ centered at a point $\theta \in \Lambda_{C'}$, and whose closure contains a point $\tilde{z} \in \tilde{Z}'$.

Let $o \in \tilde{X}$ and $E \in T_o\tilde{X}$ an hyperplane such that $\tilde{F}(o) = o$, $|Jac_E\tilde{F}(o)| = 1$, and $\Lambda_{C'} \subset E(\infty)$ where $E(\infty)$ is the topological equator associated to E.

For each $\theta \in \partial\tilde{X}$ we denote by α_θ the geodesic starting from o and such that $\alpha_\theta(+\infty) = \theta$.

Let $\theta_i \in \partial\tilde{X} - E(\infty) = \partial\tilde{V} - \partial\tilde{U}$ be a sequence converging to θ_0. By continuity, for each i large enough, the geodesic α_{θ_i} spends some time inside the horoball $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ and ends up inside $\tilde{V}$ because θ_i converges to θ_0 and θ_i belongs to $\partial\tilde{V} - \partial\tilde{U}$.

Thus α_{θ_i} eventually crosses $\tilde{Z}'$. Let $z_i \in \alpha_{\theta_i} \cap \tilde{Z}'$. As $\tilde{Z}'/C'$ is compact, there is an element $\gamma_i \in C'$ such that $z_i = \gamma_i(x_i)$ where x_i is a point in the closure $\bar{D}$ of a fundamental domain D for the action of C' on $\tilde{Z}'$. The points $\gamma_i(x_i)$ and $\gamma_i(o)$ stay at bounded distance because $dist(\gamma_i(x_i), \gamma_i(o)) = dist(x_i, o) \leq dist(o, D) + diamD$. In particular, $lim_{i\to\infty}\gamma_i(o) = \theta_0$.

We have proved the

Lemma 5.14. *Let $\theta_i \in \partial\tilde{X} - E(\infty)$ be a sequence which converges to θ_0. There exists a constant A such that for i large enough there exists $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$ and $\gamma_i \in C'$ such that $dist(z_i, \gamma_i(o)) \leq A$ and both z_i and $\gamma_i(o)$ converge to θ_0.*

Let x and y two points in $\tilde{X}$.

We define the shadow $\mathcal{O}(x, y, R) \subset \partial\tilde{X}$ of the ball $B(y, R)$ enlighted from the point x by

$$\mathcal{O}(x, y, R) = \{\alpha(+\infty)\} \tag{5.29}$$

where α runs through the set of geodesic rays starting from x and meeting $B(y, R)$.

Let $\{\mu_x\}_x$ be a family of Patterson measures associated to the discrete group C' with critical exponent $\delta' = \delta(C')$.

The following shadow lemma is due to D.Sullivan.

Lemma 5.15. [18], [13], [20]. *There exist positive constants C and R such that for any y in $\tilde{X}$, $\nu_y(\mathcal{O}(y, \gamma(y), R)) \geq Ce^{\delta' d(y,\gamma(y))}$*

Corollary 5.16. *Let z_i be defined in lemma (5.14), then we have $\mathcal{O}(o, z_i, R + A) \cap \Lambda_{C'} \neq \emptyset$ for i large enough.*

We now prove that for a good choice of θ_i, the shadow $\mathcal{O}(o, z_i, R + A)$ (with z_i associated to θ_i as in lemma 5.14) never meet $\Lambda_{C'}$ for all large i's, ie. for any $\theta \in \Lambda_{C'}$ the geodesic α_θ does not cross $B(z_i, R + A)$. We have no control on the radius R coming from the shadow lemma nor on the constant A but we will show

Proposition 5.17. *There exists a sequence $\theta_i \in \partial\tilde{X} - \Lambda_{C'}$ such that θ_i converges to θ_0 and*

$$lim_{i\to\infty} inf_{\theta\in\Lambda_{C'}} dist(z_i, \alpha_\theta) = +\infty$$

where $z_i = \tilde{Z}' \cap \alpha_{\theta_i}$ has been constructed in lemma (5.14).

Corollary 5.18. *For i large enough, $\mathcal{O}(o, z_i, R + A) \cap \Lambda_{C'} = \emptyset$.*

The corollary (5.16) and the corollary (5.18) lead to a contradiction, which ends the proof of the proposition (5.13).

The end of the paragraph is devoted to proving the proposition (5.17).

Lemma 5.19. *Let θ_i be a sequence of points in $\partial\tilde{X}$ converging to θ_0 and z_i constructed in lemma (5.14). Assume that $liminf_{i\to\infty} inf_{\theta\in\Lambda_{C'}} dist(z_i, \alpha_\theta) = C < +\infty$ then $lim_{i\to\infty} \frac{d(\theta_i,\Lambda_{C'})}{d(\theta_i,\theta_0)} = 0$.*

Proof : We first show that

$$\lim_{i\to\infty} dist(z_i, \alpha_{\theta_0}) = \infty \tag{5.30}$$

Recall that, for any $z \in \tilde{X}$ and $\theta \in \partial\tilde{X}$, $B(z, \theta)$ equals the decreasing limit as t tends to infinity of $dist(z, \alpha_\theta(t)) - dist(o, \alpha_\theta(t))$ where $\alpha_\theta(t)$ is the geodesic ray joigning o to θ. Therefore, as the points $z_i \in \tilde{Z}$ belongs to the complementary of the fixed horoball $HB(\tilde{z}, \theta_0)$, we have,

$$dist(z_i, \alpha_{\theta_0}(T_i) \geq T_i + B(\tilde{z}, \theta_0) \tag{5.31}$$

where $dist(z_i, \alpha_{\theta_0}(T_i)) = dist(z_i, \alpha_{\theta_0})$.

On the other hand, as z_i tends to θ_0, T_i tends to infinity so (5.30) is proven.

Let t_i be such that $z_i = \alpha_{\theta_i}(t_i)$. By (5.30) we have

$$\lim_{i\to\infty} dist(\alpha_{\theta_i}(t_i), \alpha_{\theta_0}(t_i)) = \infty. \tag{5.32}$$

Let u_i be such that

$$\mathrm{dist}(\alpha_{\theta_\mathrm{i}}(\mathrm{u_i}), \alpha_{\theta_0}(\mathrm{u_i})) = 1, \tag{5.33}$$

then in particular $u_i \leq t_i$ for i large enough and by the triangle inequality we have

$$dist(\alpha_{\theta_i}(t_i), \alpha_{\theta_0}(t_i)) \leq 2(t_i - u_i) + 1. \tag{5.34}$$

By (5.32) we get

$$\lim_{i\to\infty} (t_i - u_i) = +\infty. \tag{5.35}$$

Let us assume there exists a sequence $\theta'_i \in \Lambda_{C'}$ and a constant C such that

$$dist(z_i, \alpha_{\theta'_i}) \leq C < +\infty. \tag{5.36}$$

We can assume that $C \geq 1$.
Let v_i be such that

$$dist(z_i, \alpha_{\theta'_i}) = dist(z_i, \alpha_{\theta'_i}(v_i)). \tag{5.37}$$

By triangle inequality,

$$|t_i - v_i| \leq C \tag{5.38}$$

and,

$$dist(\alpha_{\theta'_i}(t_i), \alpha_{\theta_i}(t_i)) \leq 2C. \tag{5.39}$$

On the other hand, as the curvature of $\tilde{X}$ is bounded above by -1, a classical comparison theorem gives for any $t \in [0, t_i]$,

$$sinh(\frac{dist(\alpha_{\theta'_i}(t), \alpha_{\theta_i}(t))}{2}) \leq sinhC.\frac{sinht}{sinht_i}. \tag{5.40}$$

Let s_i be such that

$$dist(\alpha_{\theta'_i}(s_i), \alpha_{\theta_i}(s_i)) = 1. \tag{5.41}$$

There are two cases. Either $s_i \geq t_i$ or $s_i < t_i$. If $s_i < t_i$, we get from (5.39) and (5.40) the existence of a constant A such that for any i,

$$s_i \geq t_i - A, \tag{5.42}$$

and this inequality also holds when $s_i \geq t_i$.

¿From (5.42) we get

$$\frac{d(\theta_i, \theta_i')}{d(\theta_i, \theta_0)} = e^{-s_i+u_i} \leq e^A e^{-t_i+u_i}, \tag{5.43}$$

therefore, thanks to (5.35) we obtain

$$lim_{i\to\infty} \frac{d(\theta_i, \theta_i')}{d(\theta_i, \theta_0)} = 0. \tag{5.44}$$

which ends the proof of lemma (5.19). □

Lemma 5.20. *Let us assume that for every sequence θ_i of points in $\partial\tilde{X}$ converging to θ_0, $lim_{i\to\infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0$. Let $\lambda_k \to \infty$ be such that the sequence of spaces $(\partial\tilde{X}, \lambda_k d, \theta_0)$ converges to a space $(S, \bar{d}, 0)$ in the pointed Gromov-Hausdorff topology, then the sequence of spaces $(\Lambda_{C'}, \lambda_k d, \theta_0)$ also converges to $(S, \bar{d}, 0)$.*

Proof : Let us define

$$r(\epsilon) =: sup\{\frac{d(\theta, \Lambda_{C'})}{d(\theta, \theta_0)}, \theta \neq \theta_0, d(\theta, \theta_0) \leq \epsilon\}.$$

The assumption says that

$$lim_{\epsilon\to 0} r(\epsilon) = 0. \tag{5.45}$$

For an arbitrary metric space (Y, d) and Y' a subset of Y, let us denote $B_{(Y,d)}(y, R)$ the closed ball of (Y, d) of radius R centered at $y \in Y$, and $\mathcal{U}_\epsilon^{(Y,d)}(Y')$ the ϵ-neighborhood of Y' in (Y, d). For a metric space (Y, d) and a positive number λ, let us denote λY the rescaled space $(Y, \lambda d)$.

By definition of the function r, we have for any R,

$$B_{\lambda_k \partial\tilde{X}}(\theta_0, R) \subset \mathcal{U}_{\epsilon_k}^{\lambda_k \partial\tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k)$$

$$\subset B_{\lambda_k \partial\tilde{X}}(\theta_0, R + 2\epsilon_k). \tag{5.46}$$

where $\epsilon_k =: Rr(R/\lambda_k)$.

Let us fix $\alpha > 0$. By definition 5.7, for any $R > 0$, $\epsilon > 0$, there exist a map $f : B_{\lambda_k \partial\tilde{X}}(\theta_0, R + \alpha) \to S$ such that for $k \geq k_0$,

(i) $f(\theta_0) = 0$,

for any two points x and y in $B_{\lambda_k \partial\tilde{X}}(\theta_0, R + \alpha)$,

(ii) $|\bar{d}(f(x), f(y)) - \lambda_k d(x, y)| \leq \epsilon$,
and
(iii) $B_{(S,\bar{d})}(0, R+\alpha-\epsilon) \subset \mathcal{U}_\epsilon^{(S,\bar{d})} f(B_{\lambda_k \partial\tilde{X}}(\theta_0, R+\alpha))$.

Moreover let us prove:
(iv) $B_{(S,\bar{d})}(0, R-2\epsilon) \subset \mathcal{U}_\epsilon^{(S,\delta)} f(B_{\lambda_k \partial\tilde{X}}(\theta_0, R))$.

Indeed, let $z \in B_{(S,\bar{d})}(0, R-2\epsilon)$. By (iii), there exists $\theta \in B_{\lambda_k \partial\tilde{X}}(\theta_0, R+\alpha)$ such that $\bar{d}(z, f(\theta)) \leq \epsilon$. Since $\bar{d}(z, 0) \leq R-2\epsilon$, we thus deduce from triangle inequality $\bar{d}(f(\theta), 0) \leq R-\epsilon$, and therefore we get, thanks to (i) and (ii), $\lambda_k d(\theta, \theta_0) \leq R$. □

By (5.45), for ϵ small enough, there exists $k_1 \geq k_0$ such that for any $k \geq k_1$, then $2\epsilon_k \leq \epsilon$ and

$$B_{\lambda_k \partial\tilde{X}}(\theta_0, R+2\epsilon_k) \subset B_{\lambda_k \partial\tilde{X}}(\theta_0, R+\alpha).$$

Therefore, by (5.46) and the above properties (i),(ii),(iii) and (iv) of the map f, and the triangle inequality we get,

$$B_{(S,\bar{d})}(0, R-2\epsilon) \subset \mathcal{U}_\epsilon^{(S,\bar{d})} f(B_{\lambda_k \partial\tilde{X}}(\theta_0, R))$$

$$\subset \mathcal{U}_\epsilon^{(S,\bar{d})} f(\mathcal{U}_{\epsilon_k}^{\lambda_k \partial\tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R+\epsilon_k))$$

$$\subset \mathcal{U}_{2\epsilon+\epsilon_k}^{(S,\bar{d})} f(B_{\lambda_k \Lambda_{C'}}(\theta_0, R+\epsilon_k)).$$

About the second inclusion above let us remark that the set $\mathcal{U}_{\epsilon_k}^{\lambda_k \partial\tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R+\epsilon_k)$ is contained in $B_{\lambda_k \partial\tilde{X}}(\theta_0, R+2\epsilon_k) \subset B_{\lambda \partial\tilde{X}}(\theta_0, R+\alpha)$, so that we can apply f to this set.

¿From the above inclusions we obtain

$$B_{(S,\bar{d})}(0, R-3\epsilon) \subset \mathcal{U}_{2\epsilon+\epsilon_k}^{(S,\bar{d})} f(B_{\lambda_k \Lambda_{C'}}(\theta_0, R))$$

which implies the convergence of $(\Lambda_{C'}, \lambda_k d, \theta_0)$ to $(S, \bar{d}, 0)$.
□

Corollary 5.21. *Let us assume that for every sequence θ_i of points in $\partial\tilde{X}$ converging to θ_0 and z_i the sequence of points constructed in lemma (5.14), $liminf_{i\to\infty} inf_{\theta\in\Lambda_{C'}} dist(z_i, \alpha_\theta) < +\infty$. Let $\lambda_k \to \infty$ be such that the sequence of spaces $(\partial\tilde{X}, \lambda_k d, \theta_0)$ converges to the space $(S, \delta, 0)$ in the pointed Gromov-Hausdorff topology, then the sequence of spaces $(\Lambda_{C'}, \lambda_k d, \theta_0)$ also converges to $(S, \delta, 0)$.*

We will show now that there exist a sequence of points $\theta_1^k, \theta_2^k \in \Lambda_{C'}$ converging to θ_0, such that the mutual distances $d(\theta_1^k, \theta_2^k)$, $d(\theta_1^k, \theta_0)$, $d(\theta_2^k, \theta_0)$ is tending to zero at the same rate, and the triple $\theta_1^k, \theta_2^k, \theta_0$ can be uniformly separated by elements $\gamma_k \in C'$.

Lemma 5.22. *Assume that every weak tangent of* $(\partial\tilde{X}, d)$ *at* θ_0 *belongs to* $WT(\Lambda_{C'}, d)$*, then there exist positive constants* c*,* δ*, a sequence* ϵ_k *tending to* 0 *when* k *tends to* ∞*, a sequence* $\gamma_k \in C'$*, a sequence of points* $\theta_1^k, \theta_2^k \in \Lambda_{C'}$ *such that for* $i = 1, 2$*,*
$c^{-1}\epsilon_k \le d(\theta_1^k, \theta_2^k) \le c\epsilon_k$*,*
$c^{-1}\epsilon_k \le d(\theta_i^k, \theta_0)) \le c\epsilon_k$ *and*
$d(\gamma_k\theta_1^k, \gamma_k\theta_2^k) \ge \delta$*,* $d(\gamma_k\theta_i^k, \gamma_k\theta_0)) \ge \delta$*.*

Proof : For any $x \in \tilde{X} \cup \partial\tilde{X}$ and $y \in \tilde{X} \cup \partial\tilde{X}$ let us define $\alpha_{x,y}$ the geodesic ray joining x and y. Let $o \in \tilde{X}$ and $E \in T_o\tilde{X}$ be such that $|Jac_E\tilde{F}(o)| = 1$ and $E(\infty)$ the equator associated to E. Let $\gamma_k \in C'$ be a sequence such that $\gamma_k(o)$ converges to the point θ_0 where $\theta_0 \in \Lambda_{C'}$ is the point coming from lemma 5.12. In particular, according to that lemma, there exist a point $\tilde{z} \in \tilde{Z}'$ such that the hypersurface $\tilde{Z}'$ is contained in the complementary of the open horoball $HB(\tilde{z}, \theta_0)$. We define $D := dist(\tilde{z}, o)$. As $\tilde{Z}'$ lies outside the open horoball $HB(\tilde{z}, \theta_0)$, the points $\gamma_k(o)$ belong to the complementary of the open horoball $HB(\alpha_{\tilde{z},\theta_0}(D), \theta_0)$. By standard triangle comparison argument (comparison with the hyperbolic case) the angle $Angle(\alpha_{\gamma_k(o),\theta_0}, \alpha_{\gamma_k(o),o})$ between the two geodesic rays $\alpha_{\gamma_k(o),\theta_0}$ and $\alpha_{\gamma_k(o),o}$ satisfies :

$$lim_{k\to\infty} Angle(\alpha_{\gamma_k(o),\theta_0}, \alpha_{\gamma_k(o),o}) = 0. \tag{5.47}$$

By equivariance we have $\Lambda_{C'} \subset (\gamma_k E)(\infty)$ where $\gamma_k E \subset T_{\gamma_k(o)}\tilde{X}$. For any $v \in T\tilde{X}$ let α_v be the geodesic ray such that $\dot{\alpha}_v(0) = v$. Let us denote by u_k the unit vector in $\gamma_k E$ such that $\alpha_{u_k}(+\infty) = \theta_0$ and let us choose some $w_k \in \gamma_k E$ such that $< u_k, w_k >= 0$ (this is possible because $n - 1 \ge 2$.

We claim now that there exist $v_k \in \gamma_k E$ such that the angle between v_k and w_k is not too far from 0 or π, namely

$$| < v_k, w_k > | \ge \frac{1}{(n-1)^{1/2}}, \tag{5.48}$$

and $\alpha_{v_k}(+\infty) \in \Lambda_{C'}$ or $\alpha_{v_k}(-\infty) \in \Lambda_{C'}$.

Let us prove this claim.

According to Proposition 5.1 and to (5.11), the restriction to $\gamma_k E$ of the quadratic form $h(u) = \int DB(\gamma_k(o), \theta)(u)^2 d\mu_{\gamma_k(o)}(\theta)$ verifies

$$h_{\gamma_k E}(u) = \frac{||u||^2}{n-1}. \tag{5.49}$$

Therefore, if for all $u \in \gamma_k(E)$ such that $\alpha_u(+\infty) = \theta \in \Lambda_{C'}$ we had $| < u, w_k > | < \frac{1}{(n-1)^{1/2}}$, then one would get $h(w_k) < \frac{1}{n-1}$, which contradicts (5.49) and proves the claim.

In particular the angle between u_k and v_k is not too far from $\pi/2$ for k large enough, ie.

$$|<u_k, v_k>| \leq \left(\frac{n-2}{n-1}\right)^{1/2}, \tag{5.50}$$

and thanks to (5.47), we have for k large enough

$$|<\dot{\alpha}_{\gamma_k(0),o}(0), v_k>| \leq \left(\frac{n-\frac{3}{2}}{n-1}\right)^{1/2}. \tag{5.51}$$

Let us now assume for example that $\theta_k = \alpha_{v_k}(+\infty) \in \Lambda_{C'}$. Let us show that

$$lim_{k\to\infty} d(\theta_0, \theta_k) = 0. \tag{5.52}$$

Assume that (5.52) is not true. Then, one can assume after extracting a subsequence that θ_k converges to $\theta \neq \theta_0$. Therefore the geodesic rays $\alpha_{\gamma_k(0),o}$ and α_{v_k} would converge to the geodesics $\alpha_{\theta_0,o}$ and $\alpha_{\theta_0,\theta}$ and thus the angle $Angle(\alpha_{\gamma_k(0),o}, \alpha_{v_k})$ would converge to 0. But this would contradict (5.51).

Let us now denote $\epsilon_k =: d(\theta_k, \theta_0)$. According to (5.52), $lim_{k\to\infty}\epsilon_k = 0$. We now consider the following sequence of pointed metric space $(\partial\tilde{X}, \epsilon_k^{-1}d, \theta_0)$, a subsequence of which being converging to some metric space (S, δ), cf[]. For convenience we still denote by the same index k the subsequence. By the corollary 5.21, the sequence $(\Lambda_{C'}, \epsilon_k^{-1}d, \theta_0)$ also converges to (S, δ). According to lemma 5.8, the space S is homeomorphic to $\mathbb{R}^{n-1}$. In particular there exist a sequence of points $\theta'_k \in \Lambda_{C'}$ and a constant c such that

$$c^{-1}\epsilon_k \leq d(\theta_k, \theta'_k) \leq c\epsilon_k, \tag{5.53}$$

$$c^{-1}\epsilon_k \leq d(\theta'_k, \theta_0)) \leq c\epsilon_k. \tag{5.54}$$

Thus, the points $\theta_1^k = \theta_k$ and $\theta_2^k = \theta'_k$ satisfy the two first properties of lemma 5.22.

In order to complete the proof of lemma 5.22, we will show that the elements $\eta_k =: \gamma_k^{-1}$ uniformly separate θ_0, θ_1^k and θ_2^k.

Thanks to (5.50) the angle at $\gamma_k(o)$ between θ_1^k and θ_0 is uniformly bounded away from 0 and π and so does the angle at o between $\gamma_k^{-1}(\theta_1^k)$ and $\gamma_k^{-1}(\theta_0)$. Therefore, as the angle is Hölder-equivalent to the distance d, cf. [10], there is a constant c such that

$$d(\gamma_k^{-1}(\theta_1^k), \gamma_k^{-1}(\theta_0)) \geq c. \tag{5.55}$$

Now the cocompact group Γ acts uniformly quasi-conformally on $(\partial\tilde{X}, d)$, ([4] and [19] Theorem 5.2), and so does $C' \subset \Gamma$, therefore

$$d(\gamma_k^{-1}(\theta_1^k), \gamma_k^{-1}(\theta_2^k)) \geq c, \tag{5.56}$$

and

$$d(\gamma_k^{-1}(\theta_2^k), \gamma_k^{-1}(\theta_0)) \geq c. \tag{5.57}$$

which ends the proof of lemma 5.22. □

Proof of Proposition 5.17 :

Let us assume that for every sequence θ_i of points in $\partial\tilde{X}$ converging to θ_0, $liminf_{i\to\infty} inf_{\theta\in\Lambda_{C'}} dist(z_i, \alpha_\theta) < +\infty$, then by corollary 5.21 and lemma 5.22 there exist a positive constant c, a sequence ϵ_k tending to 0 when k tends to ∞, a sequence $\gamma_k \in C'$, a sequence of points $\theta_1^k, \theta_2^k \in \Lambda_{C'}$ such that
$c^{-1}\epsilon_k \leq d(\theta_1^k, \theta_2^k) \leq c\epsilon_k$,
$c^{-1}\epsilon_k \leq d(\theta_i^k, \theta_0)) \leq c\epsilon_k$ and
$d(\gamma_k\theta_1^k, \gamma_k\theta_2^k) \geq \delta$, $d(\gamma_k\theta_i^k, \gamma_k\theta_0)) \geq \delta$.

Applying lemma 5.9 for $\mathcal{L} = \Lambda_{C'}$ and $\lambda_k = \epsilon_k^{-1}$ we conclude that $\Lambda_{C'}$ is homeomorphic to $\partial\tilde{X}$, which is impossible because $\Lambda_{C'}$ is contained in a topological equator $E(\infty)$. □

Step 4 : C' and C are convex cocompact.

We first define convex cocompactness. For a discrete group C of isometries acting on a Cartan Hadamard manifold of negative sectional curvature with limit set Λ_C, one defines the geodesic hull $\mathcal{G}(\Lambda_C)$ of Λ_C as the set of all geodesics both ends of whose belong to Λ_C.

The geodesic hull of Λ_C is a C invariant set. One says that C is convex cocompact if $\mathcal{G}(\Lambda_C)/C$ is compact.

Lemma 5.23. *C' is convex cocompact.*

Proof : Let us denote $\pi : \tilde{X} \to \tilde{X}/C'$ the projection. Assume that C' is not convex cocompact. Then, there exist a sequence $x_n \in \mathcal{G}(C')$ such that x_n tends to infinity. In particular $dist(x_n, Z') \to +\infty$, where $Z' = \tilde{Z}'/C'$ is the compact hypersurface which separates $\tilde{X}/C'$ in two unbounded connected components. There exist lifts $\tilde{x}_n$ of x_n such that

$$\tilde{x}_n \to \theta_0 \in \Lambda_{C'} \tag{5.58}$$

$$dist(\tilde{x}_n, \tilde{Z}') = dist(\tilde{x}_n, \tilde{z}_n) \tag{5.59}$$

where $\tilde{z}_n \in \tilde{Z}'$ is bounded. Therefore there exist $\tilde{z} \in \tilde{Z}'$
such that $HB(\tilde{z}, \theta_0) \subset \tilde{U}$, where $\tilde{U}$ is one of the two connected components of $\tilde{X} - \tilde{Z}'$, the other being $\tilde{V}$.

We recall that $\mathcal{M}$, $\mathcal{N}$ are the two connected components of $\partial\tilde{X} - \Lambda_{C'}$. We also have $\partial\tilde{Z}' = \Lambda_{C'} = E(\infty)$, and after possibly replacing C' by an index two subgroup, we can assume that C' preserves $\tilde{U}$ and $\tilde{V}$.

Claim : There are the two following cases.

Either one of the two boundaries $\partial\tilde{U}$ or $\partial\tilde{V}$ is equal to $\Lambda_{C'}$ (in this case the other boundary is equal to $\partial\tilde{X}$), or $\partial\tilde{U} = \bar{\mathcal{M}}$ and $\partial\tilde{V} = \bar{\mathcal{N}}$, where $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ are the closure of $\mathcal{M}$ and $\mathcal{N}$.

Let us prove the claim. We first remark that if there exist $\theta \in \partial\tilde{U} \cap \mathcal{M}$, then $\mathcal{M} \subset \partial\tilde{U}$. Namely, let ξ be any other point in $\mathcal{M}$ and α a continuous path in $\mathcal{M}$ joining θ and ξ. Since the set $\tilde{Z}' \cup \Lambda_{C'}$ is a closed subset in $\tilde{X} \cup \partial\tilde{X}$, there exist an open connected neighborhood W of α in $\tilde{X} \cup \partial\tilde{X}$ contained in the complementary of $\tilde{Z}' \cup \Lambda_{C'}$. Therefore, as $W \cap \tilde{U} \neq \emptyset$, we have $W \cap \tilde{X} \subset \tilde{U}$ and $\xi \in \partial\tilde{U}$. Let us assume that neither $\partial\tilde{U}$ nor $\partial\tilde{V}$ is equal to $\Lambda_{C'}$. Then, each boundary $\partial\tilde{U}$ and $\partial\tilde{V}$ contains $\mathcal{M}$ or $\mathcal{N}$. But on the other hand, since the set $\tilde{Z}' \cup \Lambda_{C'}$ is closed, $(\partial\tilde{U} - \Lambda_{C'}) \cap (\partial\tilde{V} - \Lambda_{C'}) = \emptyset$ thus we have $\partial\tilde{U} = \mathcal{M}$ and $\partial\tilde{V} = \mathcal{N}$ or the other way around and the claim is proved.

Case 1 : $\partial\tilde{U} = \Lambda_{C'}$ and $\partial\tilde{V} = \partial\tilde{X}$ or the other way around.

In this case, we are in the situation of the step 3, which leads to a contradiction, cf. remark 5.11.

Case 2 : $\partial\tilde{U} = \bar{\mathcal{M}}$ and $\partial\tilde{V} = \bar{\mathcal{N}}$.

In that case, assuming C' is not convex-cocompact, there exist an open horoball $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ where $\theta_0 \in \Lambda_{C'}$, $\tilde{z} \in \tilde{Z}'$, $\partial\tilde{U} = \bar{\mathcal{M}}$ and $\partial\tilde{V} = \bar{\mathcal{N}}$. We will find a contradiction in a similar way as in case 1, ie. step 3. We consider a point $o \in \tilde{X}$ and an hyperplane $E \subset T_o\tilde{X}$ such that $|Jac_E\tilde{F}(o)| = 1$ and $\Lambda_{C'} = E(\infty)$.

Let $\theta_i \in \mathcal{N}$ be a sequence which converge to θ_0. By continuity, for i large enough, the geodesic ray α_{o,θ_i} spends some time in $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ and ends up in $\tilde{V}$ because θ_i converges to θ_0 and θ_i belongs to $\mathcal{N} = \partial\tilde{V} - \Lambda_{C'}$. Therefore, α_{o,θ_i} eventually crosses $\tilde{Z}'$. Let z_i be some point in $\tilde{Z}' \cap \alpha_{o,\theta_i}$.

We will prove the following Proposition, similar to the Proposition 5.18,

Proposition 5.24. *There exist a sequence $\theta_i \in \mathcal{N}$ such that θ_i converges to θ_0 and*

$$lim_{i\to\infty} inf_{\theta\in\Lambda_{C'}} dist(z_i, \alpha_\theta) = +\infty$$

where $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$.

Remark 5.25. *The difference between the propositions 5.24 and 5.18 is that we are looking for a sequence $\theta_i \in \mathcal{N}$ instead of $\theta_i \in \partial\tilde{X} - \Lambda_{C'}$.*

Assuming the Proposition 5.24 we find a contradiction in the same way as in step 3. Namely, as $\tilde{Z}'/C'$ is compact, the points $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$ stay at bounded distance from the C'-orbit of a fixed point, say, o, thus there exist a constant $A > 0$ and elements $\gamma_i \in C'$ such that for any i,

$$dist(z_i, \gamma_i o) \leq A. \tag{5.60}$$

¿From (5.60) and the shadow lemma 5.15, we obtain $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} \neq \emptyset$, and on the other hand, from the proposition 5.24, we have $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} = \emptyset$, which gives the contradiction. It remains to prove the Proposition 5.24.

Proof of the proposition 5.24 : We argue by contradiction, like in the proof of the proposition 5.17. Let us assume that there exist a constant $C > 0$ such that for any sequence of points $\theta_i \in \mathcal{N}$ converging to θ_0, $lim_{i\to\infty} inf_{\theta\in\Lambda_{C'}} dist(z_i, \alpha_\theta) \leq C$, then by lemma 5.19, we have for any such sequence $\theta_i \in \mathcal{N}$

$$lim_{i\to\infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0. \tag{5.61}$$

The proof of the following lemma is the same as the proof of lemma 5.20.

Lemma 5.26. *Let us assume that for any sequence $\theta_i \in \mathcal{N}$ conveging to θ_0, $lim_{i\to\infty} \frac{d(\theta_i,\Lambda_{C'})}{d(\theta_i,\theta_0)} = 0$. Let $\{\lambda_k\}$ be a sequence of positive numbers tending to $+\infty$ such that the sequence of spaces $(\partial\tilde{X}, \lambda_k d, \theta_0)$ converges to a space $(S, \delta, 0)$ in the pointed Gromov-Hausdorff topology, then, $(\mathcal{M}, \lambda_k d, \theta_0)$ also converges to $(S, \delta, 0)$.*

Proof : Since $\Lambda_{C'} \subset \bar{\mathcal{M}}$, the assumption implies that $lim_{\epsilon\to 0} r(\epsilon) = 0$ where

$$r(\epsilon) = sup\{\frac{d(\theta, \mathcal{M})}{d(\theta, \theta_0)}, \theta \neq \theta_0, \theta \in \mathcal{N}, d(\theta, \theta_0) \leq \epsilon\}$$

and the proof goes the same way as in lemma 5.20 replacing $\Lambda_{C'}$ by $\mathcal{M}$. □

Similarly to the lemma 5.22, we have the

Lemma 5.27. *Let us assume that every weak tangent of $(\partial\tilde{X}, d)$ at θ_0 belongs to $WT((\mathcal{M}, d))$. There exist positive constant c, δ, a sequence ϵ_k tending to 0 when k tends to $+\infty$, a sequence of $\gamma_k \in C'$, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{M}$ such that for $i \neq j \in \{0, 1, 2\}$,*
$c^{-1}\epsilon_k \leq d(\theta_i^k, \theta_j^k) \leq c\epsilon_k$ and
$d(\gamma_k\theta_i^k, \gamma_k\theta_i^k) \geq \delta$.

We can now end the proof of the proposition 5.24. Let us assume that there exist a constant $C > 0$ such that for every sequence θ_i of points in $\mathcal{N}$ converging to θ_0, $lim_{i\to\infty} inf_{\theta\in\Lambda_{C'}} dist(\theta_i, \alpha_\theta) \leq C$, then by (5.61), lemma 5.26 and lemma 5.27, there exist a sequence ϵ_k tending to 0 when k tends to ∞, a sequence $\gamma_k \in C'$, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{M}$ such that for $i \neq j \in \{0, 1, 2\}$,

$c^{-1}\epsilon_k \leq d(\theta_i^k, \theta_j^k) \leq c\epsilon_k$ and
$d(\gamma_k\theta_i^k, \gamma_k\theta_i^k) \geq \delta$.

Applying the lemma 5.9 for $\mathcal{L} = \mathcal{M}$ and $\lambda_k = \epsilon_k^{-1}$, we conclude that $\mathcal{M}$ is homeomorphic to $\partial\tilde{X}$, which is impossible because $\partial\tilde{X}$ is a sphere, and $\mathcal{M}$ is homeomorphic to an hemisphere. This ends the proof of the proposition 5.24. □

Corollary 5.28. *C is convex cocompact.*

Proof : The subgroup C' of C is convex cocompact and the limit sets of C' and C coincide by step 3, therefore C is convex cocompact. □.

Step 5: C preserves a copy of the $(n-1)$-dimensional hyperbolic space $\mathbb{H}^{n-1}$ totally geodesically embedded in $\tilde{X}$.

From the steps 1-4, we know that the groups C and C' are convex cocompact, and that their limit set Λ_C and $\Lambda_{C'}$ are equal to a topological equator $E(\infty)$.

Let us consider the essential hypersurface $Z' \subset \tilde{X}/C'$. We will show that there exist a minimizing current representing the class of Z' in $H_{n-1}(\tilde{X}/C', \mathbb{R})$ and that this minimizing current lifts to a totally geodesic hypersurface embedded in $\tilde{X}$. We will then show that this totally geodesic hypersurface is eventually hyperbolic.

We work in $\tilde{X}/C'$ and consider the essential hypersurface $Z' \subset \tilde{X}/C'$. We will now prove that there exist a minimal current representing the class of Z' in $H_{n-1}(\tilde{X}/C', \mathbb{R})$. Let $\{Z_k\}$ be a minimizing sequence of currents homologous to Z'. The othogonal projection onto the convex core of $\tilde{X}/C'$ is distance nonincreasing and thus volume nonincreasing. Therefore we can assume that the Z_k's are in the the convex core of $\tilde{X}/C'$, which is compact. By [12] (5.5), the sequence $\{Z_k\}$ subconverges to a minimal current Z_∞ in $\tilde{X}/C'$. By [12] (8.2), Z_∞ is a manifold with possible singularities of codimension greater than or equal to 8. By corollary (4.4) and minimality we get that $|Jac_{n-1}\tilde{F}(x)| = 1$ at every regular points $x \in Z_\infty$. We will use the fact that $|Jac_{n-1}\tilde{F}(x)| = 1$ at every regular points $x \in Z_\infty$ in order to prove that Z_∞ is a totally geodesic hypersurface.

Lemma 5.29. *Let x and y two distinct points in $\tilde{X}$ and $E_x \subset T_x\tilde{X}$, $E_y \subset T_y\tilde{X}$ be such that $Jac_{n-1}\tilde{F}'(x) = Jac_{E_x}\tilde{F}'(x) = 1$ and $Jac_{n-1}\tilde{F}'(y) = Jac_{E_y}\tilde{F}'(y) = 1$. Then, the geodesic $\alpha_{x,y}$ (resp. $\alpha_{y,x}$) joining x and y (resp. y and x) satisfies $\dot{\alpha}_{x,y}(0) \in E_x$, (resp. $\dot{\alpha}_{y,x}(0) \in E_y$). In particular, $\alpha_{x,y}(+\infty)$ and $\alpha_{y,x}(+\infty)$ belong to $\Lambda_{C'}$.*

Proof: Let S_x and S_y be the unit spheres of E_x and E_y. For any unit tangent vector $u \in T_z\tilde{X}$ at some point z, we define $\theta_u \in \partial\tilde{X}$ by $\dot{\alpha}_{z,\theta_u}(0) = u$. By step 3, $\Lambda_{C'} = E_x(\infty) = E_y(\infty)$, therefore for every $u \in S_x$, $\theta_u \in \Lambda_{C'}$ and there exist $v \in E_y$ such that $\theta_u = \theta_v$. As E_y is a vector space, θ_{-v} belongs to

$\Lambda_{C'}$ therefore there exist $w \in E_x$ such that $\theta_w = \theta_{-v}$. The map $f : S_x \to S_x$ defined by $f(u) = w$ is a continuous map. The lemma then boils down to proving that there exist $u \in S_x$ such that $f(u) = -u$ because in that case, x, y and θ_u are on the same geodesic α_{x,θ_u}.

The following properties of f are obvious.

(i) For every $u \in S_x$, $f(u) \neq u$.

(ii) $f \circ f = Id$.

So f is an involution of the sphere without fixed point and for any such map, we claim that there exist u in the sphere such that $f(u) = -u$. In order to prove the claim, we follow a very similar argument in [16], theorem 1. We argue by contradiction. Let us assume that for every $u \in S_x$, $f(u) \neq -u$. The map $g : S_x \to S_x$ defined by $g(u) = \frac{f(u)+u}{\|f(u)+u\|}$, is then well defined and continuous. Let us remark that as for every $u \in S_x$, $f(u) \neq -u$, then f is homotopic to the Identity, and so is g. Moreover by (ii) we clearly have $g \circ f = g$, thus the map g factorizes through S_x/G_f where G_f is the group generated by the involution f. By (i) f has no fixed point thus S_x/G_f is a manifold and the projection $p : S_x \to S_x/G_f$ is a degre 2 map. Therefore, the induced endomorphism g_* on $H_{n-1}(S_x, \mathbb{Z}_2)$ is trivial, which contradicts the fact that g is homotopic to the Identity. $\square$

Corollary 5.30. *Let $\mathcal{H}^{n-1} \subset \tilde{X}$ be an hypersurface with possibly non empty boundary $\partial\mathcal{H}^{n-1}$, such that for any $x \in \mathcal{H}^{n-1}$, $Jac_{n-1}\tilde{F}'(x) = Jac_{E_x}\tilde{F}'(x) = 1$, where E_x is the tangent space of $\mathcal{H}^{n-1}$ at x. Let us consider $x \in \mathcal{H}^{n-1}$ such that $dist_{\tilde{X}}(x, \partial\mathcal{H}^{n-1}) = r > 0$. Then, for any $x' \in \mathcal{H}^{n-1}$ with $dist_{\tilde{X}}(x, x') < r$, the geodesic $\alpha_{x,x'}$ joining x and x' is contained in $\mathcal{H}^{n-1}$. In particular, $\mathcal{H}^{n-1}$ is locally convex.*

Proof of the corollary: Let us fix $\theta \in \Lambda_{C'}$ and consider the vector field $\nabla B(y, \theta)$ in $\tilde{X}$. Let $x \in \mathcal{H}^{n-1}$. As $Jac_{E_x}\tilde{F}'(x) = 1$, we have by step 3 $\Lambda_{C'} = E_x(\infty)$. Then, for any $x \in \mathcal{H}^{n-1}$, $\nabla B(x, \theta)$ is tangent to $\mathcal{H}^{n-1}$, therefore the geodesic $\alpha_{x,\theta}$ satisfies $\alpha_{x,\theta}(t) \in \mathcal{H}^{n-1}$ for all $t \in [0, r)$. Let $x' \in \mathcal{H}^{n-1}$. By lemma 5.29, $\dot{\alpha}_{x,x'}(0) \in E_x$, therefore $\alpha_{x,x'} = \alpha_{x,\theta}$ and $\alpha_{x,x'}(t) \in \mathcal{H}^{n-1}$ for all $t \in [0, r)$. $\square$

We now prove that Z_∞ is a totally geodesic hypersurface in $\tilde{X}/C'$. Let us recall that Z_∞ is a manifold which is smooth except at a singular subset of codimension at least 7. Let us consider a lift $\tilde{Z}_\infty \subset \tilde{X}$ of Z_∞ and denote $\tilde{Z}_\infty^{reg}$ (resp. $\tilde{Z}_\infty^{sing}$) the set of regular (resp.) singular points of $\tilde{Z}_\infty$.

Lemma 5.31. *$\tilde{Z}_\infty$ is a totally geodesic hypersurface in $\tilde{X}$.*

Proof: Let us consider a regular point $x \in \tilde{Z}_\infty^{reg}$. We shall show that for every point $x' \in \tilde{Z}_\infty^{reg}$ the geodesic segment joining x and x' is contained in $\tilde{Z}_\infty$, and as the set of regular points is dense in $\tilde{Z}_\infty$ (as the complementary of a subset of codimension at least 8), this will show that $\tilde{Z}_\infty$ is totally geodesic.

We claim that there exist a sequence $y_k \in \tilde{Z}_\infty^{reg}$ such that $\lim_{k\to\infty} y_k = x'$ and the geodesic segment joining x and y_k is contained in $\tilde{Z}_\infty$.

The claim immediately implies that the geodesic segment joining x and x' is contained in $\tilde{Z}_\infty$.

Let us prove the claim.

For $y \in \tilde{Z}_\infty^{reg}$ we consider $\alpha_{x,y}$ the geodesic joining x and y and define

$$t_y = inf\{t > 0, \alpha_{x,y}(t) \notin \tilde{Z}_\infty\} \tag{5.62}$$

As x is a regular point, by corollary 5.30, there exist $\epsilon > 0$ such that $t_y > \epsilon$.

In order to prove the claim, we argue by contradiction. Let us assume that there exist $r > 0$ such that for any $y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty^{reg}$, $t_y < dist(x, y)$. By corollary 5.30 applied to $\tilde{Z}_\infty^{reg}$, we have $\alpha_{x,y}(t_y) \in \tilde{Z}_\infty^{sing}$. As the set of regular points is an open subset of $\tilde{Z}_\infty$, if r is small enough we have $B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty^{reg} = B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty$. We choose such an r and we consider the set S of all singular points contained in the union of all geodesic segments joining x to a point $y \in B_{\tilde{X}}(x', r) \cap Z_\infty$. Let us consider the map defined on S by

$$p(y) = \alpha_{x,y}(\epsilon).$$

As we already saw, for any $y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty$, we have $t_y > \epsilon$, therefore the map p is distance decreasing and by assumption p is surjective onto an open subset of the sphere and $p(S)$ is homeomorphic to an open subset of $\mathbb{R}^{n-1}$, therefore the Hausdorff dimension of S is greater than or equal to $n-1$, which contradicts the fact that the singular set has codimension at least 8 in $\tilde{Z}_\infty$. $\square$

The totally geodesic hypersurface $\tilde{Z}_\infty \subset \tilde{X}$ is preserved by C, and $\tilde{Z}_\infty/C$ is of minimal volume in its homology class.

Let us prove that $\tilde{Z}_\infty$ is isometric to the hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}}$.

Lemma 5.32. *$\tilde{Z}_\infty$ is isometric to the hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}}$.*

Proof :

As $\tilde{Z}_\infty/C$ is of minimal volume in its homology class, we have by Proposition 5.1, for all $x \in \tilde{Z}_\infty$, $Jac_{E_x}\tilde{F}(x) = 1$ and $\tilde{F}(x) = x$, where E_x is the tangent space of $\tilde{Z}_\infty$ at x. Moreover, we saw in the proof of proposition 5.1 that

$$H = \frac{1}{n-1} Id_{D\tilde{F}(x)(E_x)} = \frac{1}{n-1} Id_{E_x},$$

therefore we get from (4.11) and $\tilde{F}(x) = x$, that for all $u, v \in T_x\tilde{Z}_\infty$,

$$\int_{\partial\tilde{X}} [DdB_{(x,\theta)}(u, v) + DB_{(x,\theta)}(u)DB_{(x,\theta)}(v)]d\nu_x(\theta)$$

$$= \tilde{g}(u, v) \tag{5.63}$$

where $\tilde{g}$ is the metric on $\tilde{X}$. As $\tilde{Z}_\infty$ is totally geodesic, the relation (5.62) remains true with the Busemann function $B^{\tilde{Z}_\infty}$ of $\tilde{Z}_\infty$ instead of the Busemann function B of $\tilde{X}$:

$$\int_{\partial\tilde{X}} [DdB^{\tilde{Z}_\infty}_{(x,\theta)}(u,v) + DB^{\tilde{Z}_\infty}_{(x,\theta)}(u)DB^{\tilde{Z}_\infty}_{(x,\theta)}(v)$$

$$= \tilde{g}(u,v). \tag{5.64}$$

On the other hand, as $\tilde{Z}_\infty$ is totally geodesic, its sectional curvature is less than or equal to -1, thus by Rauch comparison theorem, we have

$$DdB^{\tilde{Z}_\infty}(x,\theta) + DB^{\tilde{Z}_\infty}(x,\theta) \otimes DB^{\tilde{Z}_\infty}(x,\theta) \geq \tilde{g}_{\tilde{Z}_\infty} \tag{5.65}$$

for all $\theta \in \partial\tilde{Z}_\infty = \Lambda_C$, where $\tilde{g}_{\tilde{Z}_\infty}$ is the restriction of $\tilde{g}$ to $\tilde{Z}_\infty$.

As the support of the measure ν_x is $\partial\tilde{Z}_\infty = \Lambda_C$ (by convex cocompactness of C) and the Busemann function is continuous, we get from (5.64) and (5.65) that for all $x \in \tilde{Z}_\infty$ and all $\theta \in \tilde{Z}_\infty$

$$DdB^{\tilde{Z}_\infty}(x,\theta) + DB^{\tilde{Z}_\infty}(x,\theta) \otimes DB^{\tilde{Z}_\infty}(x,\theta)$$

$$= \tilde{g}_{\tilde{Z}_\infty}(x). \tag{5.66}$$

and this last relation is characteristic of the hyperbolic space. □

Step 6: Conclusion

So far we have shown that C preserves a totally geodesic copy of the hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}} \subset \tilde{X}$ such that $\mathbb{H}^{n-1}_{\mathbb{R}}/C$ is compact.

Our goal now is to show that $Y =: \mathbb{H}^{n-1}_{\mathbb{R}}/C$ injects diffeomorphically in $X = \tilde{X}/\Gamma$ and separates X in two connected components R and S such that $A = \pi_1(R)$ and $B = \pi_1(S)$.

In order to do this, we will consider the Γ orbit of $\mathbb{H}^{n-1}_{\mathbb{R}}$ in $\tilde{X}$ and the two connected components U and V of $\tilde{X} - \Gamma\mathbb{H}^{n-1}_{\mathbb{R}}$ which are adjacent to $\mathbb{H}^{n-1}_{\mathbb{R}}$. The stabilizers $\bar{A}$, $\bar{B}$ and $\bar{C}$ of U, V and $\mathbb{H}^{n-1}_{\mathbb{R}}$ contain respectively A, B and C and the hypersurface $\mathbb{H}^{n-1}_{\mathbb{R}}/C$ injects in $X = \tilde{X}/\Gamma$ and separates X in two connected components R and S such that $\pi_1(R) = \bar{A}$ and $\pi_1(S) = \bar{B}$. We then show that $\bar{C} = C$, $\bar{A} = A$ and $\bar{B} = B$.

Let $\bar{C}$ be the stabilizer of $\mathbb{H}^{n-1}$, namely $\bar{C} = \{\gamma \in \Gamma, \gamma\mathbb{H}^{n-1} = \mathbb{H}^{n-1}\}$. We have $C \subset \bar{C}$ and as $\mathbb{H}^{n-1}/C$ is compact, so is $\mathbb{H}^{n-1}/\bar{C}$ and thus $[\bar{C} : C] < \infty$.

Let $p : \tilde{X}/C \to X = \tilde{X}/\Gamma$ and $\bar{p} : \tilde{X}/\bar{C} \to X = \tilde{X}/\Gamma$ the natural projections. We now show that the restriction of p to $\mathbb{H}^{n-1}/C$ is an embedding, thus $Y := p(\mathbb{H}^{n-1}/C)$ is a compact totally geodesic hypersurface of X.

In the section 2, we constructed a C-invariant hypersurface $\tilde{Z} \subset \tilde{X}$ such that $Z = \tilde{Z}/C \subset \tilde{X}/C$ is compact. The hypersurface is defined as $\tilde{Z} = \tilde{f}^{-1}(t_0)$ where $\tilde{f} : \tilde{X} \to T$ is an equivariant map onto the Bass-Serre tree associated to the amalgamation $A *_C B$ and t_0 belongs to that edge of T which is fixed by C.

Let us first show two lemmas.

Lemma 5.33. *The restriction of p to $\tilde{Z}/C$ is an embedding into $X = \tilde{X}/\Gamma$.*

Proof : Let $\gamma \in \Gamma$, z, z' in $\tilde{Z}$ such that $z' = \gamma z$. By equivariance,

$$\tilde{f}(\gamma z) = \gamma \tilde{f}(z) = \gamma t_0 = \tilde{f}(z') = t_0,$$

thus $\gamma \in C$.$\square$

Lemma 5.34. *The restriction of $\bar{p}$ to $\mathbb{H}^{n-1}/\bar{C}$ is an embedding into $X = \tilde{X}/\Gamma$.*

Proof : Let us assume that there is a $\gamma \in \Gamma - \bar{C}$ such that $\gamma\mathbb{H}^{n-1} \cap \mathbb{H}^{n-1} \neq \emptyset$ and choose an $x \in \gamma\mathbb{H}^{n-1} \cap \mathbb{H}^{n-1}$. As $\gamma \notin \bar{C}$, there exist $u \in T_x\gamma\mathbb{H}^{n-1} - T_x\mathbb{H}^{n-1}$. We consider c_u the geodesic ray such that $\dot{c}_u(0) = u$. We know that $\tilde{Z}$ is contained in an ϵ-neighbourhood $\mathcal{U}_\epsilon\mathbb{H}^{n-1}$ of $\mathbb{H}^{n-1}$. The ϵ-neighbourhood $\mathcal{U}_\epsilon\mathbb{H}^{n-1}$ of $\mathbb{H}^{n-1}$ separates $\tilde{X}$ in two connected components U and V and for $t > 0$ large enough, we have, say, $c_u(t) \in U$ and $c_u(-t) \in V$.

Let $\tilde{Z}'$ be the connected component of $\tilde{Z}$ that we constructed at the end of section 2, whose stabilizer (or an index two subgroup of it) C' is such that $\tilde{Z}'/C'$ separates $\tilde{X}/C'$ in two unbounded connected components U'/C' and V'/C' where U' and V' are the two connected components of $\tilde{X} - \tilde{Z}'$.

We claim that $U \subset U'$ and $V \subset V'$ or the other way around. Indeed if not, U and V would be both contained in, say, U'. But in that case, V' would be contained in $\mathcal{U}_\epsilon H^{n-1}$ and therefore V'/C' would be bounded, which is a contradiction.

As $\gamma\tilde{Z}$ lies in the ϵ neighborhood of $\gamma\mathbb{H}^{n-1}$, there exist sequences z_k, z'_k in $\gamma\tilde{Z}$ such that $dist(z_k, c_u(k)) \leq \epsilon$ and $dist(z'_k, c_u(-k)) \leq \epsilon$. By proposition 5.13 and lemma 5.23, C' also acts cocompactly on $\mathbb{H}^{n-1}$, thus C' is of finite index in C, and therefore there are finitely many connected components of $\tilde{Z}$ and the same holds for $\gamma\tilde{Z}$. We thus can assume that the z_k's and z'_k's belong to a single connected component of $\gamma\tilde{Z}$. Let us consider a continuous path $\alpha \subset \gamma\tilde{Z}$ joining z_k and z'_k.

By construction the distance between $c_u(k)$ [resp. $c_u(-k)$] and $\mathbb{H}^{n-1}$ tends to infinity and thus, for k large enough, $z_k \in U$ and $z'_k \in V$ or the other way around. By the claim, we then have $z_k \in U'$ and $z'_k \in V'$, therefore the path α has to cross $\tilde{Z}'$ which contradicts the lemma (5.29) and ends the proof of the lemma 5.30. $\square$

As we already saw, $\tilde{Z}$ has finitely many connected components, and so does $\tilde{X} - \tilde{Z}$. Let us write $\{W_j\}_{j=1,..,m}$ the connected components of $\tilde{X} - \tilde{Z}$. As C acts cocompactly on $\tilde{Z}$ and $\mathbb{H}^{n-1}$ there exist $\epsilon > 0$ such that $\mathbb{H}^{n-1} \subset$

$\mathcal{U}_\epsilon \tilde{Z}$ and $\tilde{Z} \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1}$. Moreover $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$ separates $\tilde{X}$ in two connected components U and V.

Lemma 5.35. *Let us consider ϵ such that $\tilde{Z} \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1}$ and U and V the two connected components of $\tilde{X} - \mathcal{U}_\epsilon \mathbb{H}^{n-1}$. There are two distinct connected components W_1 and W_2 of $\tilde{X} - \tilde{Z}$ such that $U \subset W_1$ and $V \subset W_2$. Moreover, $\tilde{f}(W_1) \subset \tilde{T}_1$ and $\tilde{f}(W_2) \subset \tilde{T}_2$, where $\tilde{T}_1$ and $\tilde{T}_2$ are the two connected components of $\tilde{T} - \{t_0\}$.*

Proof : We argue by contradiction. Let us assume that U and V are contained in the same connected component W_1 of $\tilde{X} - \tilde{Z}$. Then, all other components W_j, $j \neq 1$, satisfy $W_j \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1} \subset \mathcal{U}_{2\epsilon} \tilde{Z}$. Therefore, as C acts cocompactly on $\mathcal{U}_{2\epsilon} \tilde{Z}$, there exist a constant D such that for any $j \neq 1$, $max_{w \in W_j} dist_{\tilde{T}}(\tilde{f}(w), t_0) \leq D$. Thus, $\tilde{f}(W_1)$ is contained in one connected component of $\tilde{T} - \{t_0\}$ and $\tilde{f}(\cup_{j \neq 1} W_j)$, contained in the ball $B_{\tilde{T}}(t_0, D)$ of $\tilde{T}$ of radius D centered at t_0, is bounded. This is clearly impossible because $\tilde{T} - \{t_0\}$ has two unbounded connected components and $\tilde{f}$ is onto. □

Let us denote $\mathcal{A} = A\mathbb{H}^{n-1}$ the A-orbit of the C-invariant totally geodesic copy of the real hyperbolic space $\mathbb{H}^{n-1}$, and $\bar{A}$ the stabilizer of $\mathcal{A}$, ie. $\bar{A} = \{\gamma \in \Gamma, \gamma\mathcal{A} = \mathcal{A}\}$. We define in a similar way $\mathcal{B} = B\mathbb{H}^{n-1}$ and $\bar{B} = \{\gamma \in \Gamma, \gamma\mathcal{B} = \mathcal{B}\}$.

Let us recall that $\bar{C}$ is the stabilizer of $\mathbb{H}^{n-1}$ in Γ. We now prove the following

Lemma 5.36. *We have $\bar{A} = A\bar{C}$ and $\bar{B} = B\bar{C}$. Moreover, $\bar{A}$ and $\bar{B}$ are charactrized by $\bar{A} = \{\gamma \in \Gamma, \gamma\mathbb{H}^{n-1} \in \mathcal{A}\}$ and $\bar{B} = \{\gamma \in \Gamma, \gamma\mathbb{H}^{n-1} \in \mathcal{B}\}$.*

Proof : Let $\gamma' \in \bar{A}$, then $\gamma'\mathbb{H}^{n-1} \in \mathcal{A}$ and thus there exist $\gamma \in A$ such that $\gamma'\mathbb{H}^{n-1} = \gamma\mathbb{H}^{n-1}$, therefore $\gamma^{-1}\gamma' \in \bar{C}$, which proves the first part of the lemma.

Let us prove the second part of the lemma.

Let $\gamma' \in \Gamma$ be such that $\gamma'\mathbb{H}^{n-1} \in \mathcal{A}$. Then there exist $\gamma \in A$ such that $\gamma'\mathbb{H}^{n-1} = \gamma\mathbb{H}^{n-1}$, thus $\gamma^{-1}\gamma' \in \bar{C}$ and therefore $\gamma' \in \gamma\bar{C} \subset A\bar{C} = \bar{A}$. This proves one inclusion, the other inclusion being obvious. □

For each $\gamma \in \Gamma$, $\gamma\mathbb{H}^{n-1}$ separates $\tilde{X}$ in two connected components U_γ and V_γ.

Let us now prove the following lemma.

Lemma 5.37. *(i) Let $\gamma \in A$, [resp. $\gamma \in B$]. Then, we have $\mathcal{A} - \{\gamma\mathbb{H}^{n-1}\} \subset U_\gamma$ or $\mathcal{A} - \{\gamma\mathbb{H}^{n-1}\} \subset V_\gamma$, [resp. $\mathcal{B} - \{\gamma\mathbb{H}^{n-1}\} \subset U_\gamma$ or $\mathcal{B} - \{\gamma\mathbb{H}^{n-1}\} \subset V_\gamma$.]*

(ii) Let γ be an element of $\Gamma - \bar{A}$, [resp. $\Gamma - \bar{B}$]. Then $\mathcal{A} \subset U_\gamma$ or $\mathcal{A} \subset V_\gamma$, [resp. $\mathcal{B} \subset U_\gamma$ or $\mathcal{B} \subset V_\gamma$].

Proof : (i) We argue by contradiction. Let us consider $\gamma\mathbb{H}^{n-1}$, $\gamma'\mathbb{H}^{n-1}$ and $\gamma''\mathbb{H}^{n-1}$ three distinct elements in $\mathcal{A}$ such that $\gamma'\mathbb{H}^{n-1} \subset U_\gamma$ and $\gamma''\mathbb{H}^{n-1} \subset V_\gamma$. By equivariance we can assume γ is the identity. Let us recall that U and V are the two connected components of $\tilde{X} - \mathcal{U}_\epsilon \mathbb{H}^{n-1}$.

We then have $\gamma'\mathbb{H}^{n-1} \cap U \neq \emptyset$ and $\gamma''\mathbb{H}^{n-1} \cap V \neq \emptyset$, which implies $\gamma'\tilde{Z} \cap U \neq \emptyset$ and $\gamma''\tilde{Z} \cap V \neq \emptyset$.

By lemma (5.31), $U \subset W_1$ and $V \subset W_2$ where W_1 and W_2 are two connected components of $\tilde{X} - \tilde{Z}$ and $\tilde{f}(U) \subset \tilde{T}_1$ and $\tilde{f}(V) \subset \tilde{T}_2$, therefore, $\tilde{f}(U)$ contains $\gamma' t_0 \in \tilde{T}_1$ and $\tilde{f}(V)$ contains $\gamma'' t_0 \in \tilde{T}_2$. This is impossible because for all elements γ' and γ'' in A, $\gamma' t_0$ and $\gamma'' t_0$ belong to the same connected component of $\tilde{T} - \{t_0\}$.

(ii) Let us consider $\gamma \in \Gamma - \bar{A}$. We argue by contradiction. Let us assume there exist γ, γ' in A such that

$$\gamma'\mathbb{H}^{n-1} \subset U_\gamma$$

$$\gamma''\mathbb{H}^{n-1} \subset V_\gamma \tag{5.67}$$

Let $\epsilon > 0$ such that $\tilde{Z} \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1}$ and U and V the connected component of $\tilde{X} - \mathcal{U}_\epsilon \mathbb{H}^{n-1}$. By lemma (5.31) we have $\tilde{f}(\gamma U) \subset \gamma \tilde{T}_1$ and $\tilde{f}(\gamma V) \subset \gamma \tilde{T}_2$, where $\gamma \tilde{T}_1$ and $\gamma \tilde{T}_2$ are the two connected components of $\tilde{T} - \{\gamma t_0\}$. By assumption (5.62), we have

$\gamma'\mathbb{H}^{n-1} \cap \gamma U \neq \emptyset$ and $\gamma''\mathbb{H}^{n-1} \cap \gamma V \neq \emptyset$, which implies $\gamma'\tilde{Z} \cap \gamma U \neq \emptyset$ and $\gamma''\tilde{Z} \cap \gamma V \neq \emptyset$, therefore $\gamma' t_0 \in \gamma \tilde{T}_1$ and $\gamma'' t_0 \in \gamma \tilde{T}_2$, which is impossible because in the tree $\tilde{T}$, the points $\gamma' t_0$ and $\gamma'' t_0$ belong to two adjacent edges.

□

By lemma (5.33) (i), for every γ in $\bar{A}$, [resp. $\bar{B}$], we can define U_γ as the connected component of $\tilde{X} - \mathbb{H}^{n-1}$ which contains all $\gamma'\mathbb{H}^{n-1}$ for all γ' in $\bar{A}$, [resp. $\bar{B}$], and $\gamma'\mathbb{H}^{n-1} \neq \gamma\mathbb{H}^{n-1}$.

Let us define

$$U_A := \cap_{\gamma \in A} U_\gamma. \tag{5.68}$$

By definition, U_A [resp. U_B]is a convex set in $\tilde{X}$ whose boundary is the collection $\mathcal{A}$, [resp. $\mathcal{B}$], of $\gamma\mathbb{H}^{n-1}$, γ in $\bar{A}$ [resp. $\bar{B}$] and by lemma (5.33) (ii), U_A and U_B are two disjoint connected components of $\tilde{X} - \Gamma\mathbb{H}^{n-1}$.

In fact, U_A [resp. U_B], is the convex hull of $\mathcal{A}$, [resp. $\mathcal{B}$], and $\bar{A}$, [resp. $\bar{B}$], is the stabilizer of U_A, [resp. U_B].

Lemma 5.38. *The closures of U_A and U_B intersect along $\mathbb{H}^{n-1}$ and $\bar{A} \cap \bar{B} = \bar{C}$. Moreover, no element of Γ sends U_A on U_B nor the other way around.*

Proof : The convex set U_A is the intersection of open half spaces U_γ, $\gamma \in \bar{A}$, and is delimited by the disjoint union of hyperplanes $\gamma\mathbb{H}^{n-1}$, for some $\gamma \in \bar{A}$. The same is true for U_B and as $U_A \cap U_B = \emptyset$, the closures of U_A and U_B can intersect only along one of the connected components of their boundaries, thus along $\mathbb{H}^{n-1}$ which is obviously in both closures. This proves the first part of the lemma, let us prove the second part. By lemma 5.32, $\bar{C} \subset \bar{A} \cap \bar{B}$. Conversely, let us take $\gamma \in \bar{A} \cap \bar{B}$, then γ preserves the closures of U_A and U_B, thus it preserves their intersection $\mathbb{H}^{n-1}$, and therefore $\gamma \in \bar{C}$.

Let us prove the last part of the lemma. Let γ be an element such that $\gamma U_A = U_B$. As $\mathbb{H}^{n-1}$ is one component of the boundary $\mathcal{B}$ of U_B, there exist one component $\gamma'\mathbb{H}^{n-1} \in \mathcal{A}$, γ' being in $\bar{A}$, such that $\gamma(\gamma'\mathbb{H}^{n-1}) = \mathbb{H}^{n-1}$. Therefore, $\gamma\gamma' \in \bar{C}$, thus $\gamma \in \bar{A}$. The same argument yields $\gamma^{-1} \in \bar{B}$, so $\gamma \in \bar{A} \cap \bar{B} = \bar{C}$ and γ preserves U_A and U_B, which contradicts our choice of γ.

□

The Γ-orbit of the closure of $U_A \cup U_B$ covers $\tilde{X}$. Let us construct a tree $\bar{T}$ embedded in $\tilde{X}$ in the following way: the set of vertices is the set of the connected components of $\tilde{X} - \Gamma\mathbb{H}^{n-1}$ and two vertices are joined by an edge if the boundaries of their corresponding connected components intersect non trivially in $\tilde{X}$. By construction Γ acts on $\bar{T}$, the stabilizers of the vertices a and b corresponding to U_A and U_B are $\bar{A}$ and $\bar{B}$, the stabilizer of the edge between a and b is $\bar{C}$ and a fundamental domain for this action is the segment joining a and b. By [14], I, 4, Theorem 6, the group Γ is the amalgamated product of $\bar{A}$ and $\bar{B}$ over $\bar{C}$.

We now claim that $\bar{A} = A$, $\bar{B} = B$ and $\bar{C} = C$.

As A, B and C are subgroups of $\bar{A}$, $\bar{B}$, and $\bar{C}$, the corresponding Mayer-Vietoris sequences of $A *_C B$ and $\bar{A} *_{\bar{C}} \bar{B}$ are related by the following commutative diagram

$$\begin{array}{ccccc} H_n(A,\mathbb{R}) \oplus H_n(B,\mathbb{R}) & \longrightarrow & H_n(\Gamma,\mathbb{R}) & \longrightarrow & H_{n-1}(C,\mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow \\ H_n(\bar{A},\mathbb{R}) \oplus H_n(\bar{B},\mathbb{R}) & \longrightarrow & H_n(\Gamma,\mathbb{R}) & \longrightarrow & H_{n-1}(\bar{C},\mathbb{R}). \end{array}$$

We know that the index $[\bar{C} : C]$ is finite, and by the lemma 5.32, the indices of A, B, and C in $\bar{A}$, $\bar{B}$ and $\bar{C}$ are finite and equal. On the other hand, the indices $[\Gamma : A]$ and $[\Gamma : B]$ are infinite by assumption, thus the previous diagram becomes

$$\begin{array}{ccccc} 0 & \longrightarrow & H_n(\Gamma,\mathbb{R}) & \longrightarrow & H_{n-1}(C,\mathbb{R}) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_n(\Gamma,\mathbb{R}) & \longrightarrow & H_{n-1}(\bar{C},\mathbb{R}). \end{array}$$

Moreover, the map $H_n(\Gamma,\mathbb{R}) \to H_{n-1}(\bar{C},\mathbb{R})$ is bijective. Namely, the injectivity comes from the above diagram and the surjectivity from the fact that that the hypersurface $\mathbb{H}^{n-1}/\bar{C}$ bounds in $\mathbb{H}^n/A$ and $\mathbb{H}^n/B$ so that the map $H_{n-1}(C,\mathbb{R}) \to H_{n-1}(A,\mathbb{R}) \oplus H_{n-1}(B,\mathbb{R})$ is trivial. Therefore the index $[\bar{C} : C] = 1$ and we get $\bar{A} = A$, $\bar{B} = B$ and $\bar{C} = C$. □

6. Proof of the theorems 1.5 and 1.6.

The proof of theorem 1.5 is exactly the same as the proof of theorem 1.2. The actions of Γ on $\tilde{(X)}$ and T give rise to a continuous Γ-equivariant map $\tilde{f} : \tilde{X} \to T$. Like in section 2, we build an hypersurface $\tilde{f}^{-1}(t_0)$ where t_0 is a regular value of $\tilde{f}$ belonging the interior of an edge. As the edge separates the tree in two unbounded components, the section 2 applies and we get a subgroup C' of C, and an hypersurface $\tilde{Z}' \subset \tilde{X}/C'$ which is essential. Now, if the action of Γ is minimal, every edge separates T in two unbounded components. □

7. Appendix

The goal of this section is to give a proof of lemma 5.9. This lemma is contained in lemma 2.1, 5.1 and 5.2 of [4], but our situation being not exactly the same, we reproduce it down here for sake of completness.

Let us restate the lemma 5.9.

Lemma 7.1. *Let $\mathcal{L} \subset \partial\tilde{X}$ be a closed C'-invariant subset and $\theta_0 \in \mathcal{L}$. We assume that there exist a sequence of positive real numbers $\lambda_k \to \infty$ such that the sequence of pointed metric spaces $(\mathcal{L}, \lambda_k d, \theta_0)$ converges in the pointed Gromov-Hausdorff topology to $(S, \delta, 0)$ where $(S, \delta, 0)$ is a weak tangent of $(\partial\tilde{X}, d)$. We also assume that there exist positive constants C and δ, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{L}$ and a sequence of elements $\gamma_k \in C'$ such that $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C\lambda_k^{-1}$ and $d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \delta$ for all $0 \leq i \neq j \leq 2$. Then, $\mathcal{L}$ is homeomorphic to the one point compactification $\hat{S}$ of S. In particular $\mathcal{L}$ is homeomorphic to $\partial\tilde{X}$.*

We first give a definition of pointed Hausdorff-Gromov convergence which is equivalent to the definition 5.7. We follow [4], paragraph 4.

A sequence of metric spaces (Z_k, d_k, z_k) converges to the metric space $(S, \delta, 0)$ if for every $R > 0$, and every $\epsilon > 0$, there exist an integer N, a subset $D \subset B_S(0, R)$, subsets $D_k \subset B_{Z_k}(z_k, R)$ and bijections $f_k : D_k \to D$ such that for $k \geq N$,

(i) $f_k(z_k) = 0$,
(ii) the set D is ϵ-dense in $B_S(0, R)$, and the sets D_k are ϵ-dense in $B_{Z_k}(z_k, R)$,
(iii) $|d_{Z_k}(x, y) - d_Z(f_k(x), f_k(y))| < \epsilon$,
where x, y belong to D_k.

Let us describe now the lemmas 2.1 and 5.1 following [4].

For a metric space (Z, d) the cross ratio of four points $\{z_i\}$, $i = 1, ...4$, is the quantity

$$[z_1, z_2, z_3, z_4] := \frac{d(z_1, z_3)d(z_2, z_4)}{d(z_1, z_4)d(z_2, z_3)} \tag{7.1}$$

Given two metric spaces X and Y, an homeomorphism $\eta : [0, \infty) \to [0, \infty)$, and an injective map $f : X \to Y$, we say that f is an η-quasi-Möbius map if for any four points $\{x_i\}$, $i = 1, .., 4$, in X, we have

$$[f(x_1), f(x_2), f(x_3), f(x_4)] \leq \eta([x_1, x_2, x_3, x_4]). \tag{7.2}$$

For example, any discrete cocompact group of isometries of $\tilde{X}$, where $\tilde{X}$ is a Cartan-Hadamard manifold with sectional curvature $K \leq -1$, is acting on the ideal boundary $(\partial\tilde{X}, d)$ endowed with the Gromov distance by η-quasi-Möbius transformations for some η.

Lemma 7.2 ([4], Lemma 2.1). *Let (X, d_X) and (Y, d_Y) be two compact metric spaces, and for any integer k, $g_k : \tilde{D}_k \to Y$ an η-quasi-Möbius map defined on a subset $\tilde{D}_k$ of X. We assume that the Hausdorff distance between $\tilde{D}_k$ and X satisfies*

$$lim_{k\to\infty} dist_H(\tilde{D}_k, X) = 0$$

and that for any integer k, there exist points (x_1^k, x_2^k, x_3^k) in D_k and (y_1^k, y_2^k, y_3^k) in Y, such that $g_k(x_i^k) = y_i^k$ for $i \in \{1, 2, 3\}$, $d_X(x_i^k, x_j^k) \geq \delta$ and $d_Y(y_i^k, y_j^k) \geq \delta$ for $i, j \in \{1, 2, 3\}, i \neq j$, where δ is independant of k. Then a subsequence of g_k converges uniformly to a quasi-Möbius map $f : X \to Y$, ie. $lim_{k_j\to\infty} dist_H(g_{k_j}, f|_{\tilde{D}_{k_j}}) = 0$. If in addition, we suppose that

$$lim_{k\to\infty} dist_H(g_k(\tilde{D}_k), Y) = 0,$$

then the sequence $\{g_{k_j}\}$ converges uniformly to a quasi-Möbius homeomorphism $f : X \to Y$.

Before stating the second lemma, let us define a metric space Z to be uniformly perfect if there exist a constant $\lambda \geq 1$ such that for every $z \in Z$ and $0 < R < diam Z$, we have $\bar{B}(z, R) - B(z, \frac{R}{\lambda}) \neq \emptyset$.

Lemma 7.3 ([4], lemma 5.1). *Let Z be a compact uniformly perfect metric space and G an η-quasi-Möbius action on Z. Suppose that for each integer k we are given a set D_k in a ball $B_k = B(z, R_k) \subset Z$ that is $(\epsilon_k R_k)$-dense in B_k, where $\epsilon_k > 0$, distinct points $x_1^k, x_2^k, x_3^k \in B(z, \lambda_k R_k)$, where $\lambda_k > 0$, with*

$$d_Z(x_i^k, x_j^k) \geq \delta_k R_k$$

for $i, j \in \{1, 2, 3\}, i \neq j$, where $\delta_k > 0$, and groups elements $\gamma_k \in G$ such that for $y_i^k := \gamma_k(x_i^k)$ we have,

$$d_Z(y_i^k, y_j^k) \geq \delta'$$

for $i, j \in \{1, 2, 3\}, i \neq j$, where δ' is independant of k. Let $D'_k = \gamma_k(D_k)$, and suppose that $\lambda_k \to 0$ when $k \to \infty$, and the sequence $\frac{\epsilon_k}{\delta_k^2}$ is bounded. Then $lim_{k\to\infty} dist_H(D'_k, Z) = 0$.

Let us go back to the proof of lemma 6.1. By definition of convergence, there exist a subsequence of $\{\lambda_k\}$, which we still denote by $\{\lambda_k\}$, subsets $\tilde{D}_k \subset B_S(0, k)$, $D_k \subset B_{\lambda_k \mathcal{L}}(\theta_0, k)$, where $\tilde{D}_k$ and D_k are minimal $1/k$-dense subsets of $B_S(0,k)$ and $B_{(\mathcal{L},\lambda_k d)}(\theta_0,k)$, and bijections $f_k : \tilde{D}_k \to D_k$ such that for all $x, y \in \tilde{D}_k$,

$$\frac{1}{2}\delta(x,y) \leq \lambda_k d(f_k(x), f_k(y)) \leq 2\delta(x,y), \tag{7.3}$$

cf. [4], (5.4).

We can suppose that the points $\theta_0^k := \theta_0$, θ_1^k, and θ_2^k in lemma 6.1 belong to the set D_k. By assumption there exist elements $\gamma_k \in C'$ and a constant δ such that

$$d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \delta \tag{7.4}$$

for all $i, j \in \{0,1,2\}$.

The lemma 6.1 is a direct consequence of the lemma 6.2 applied to $(X, d_X) = (\hat{S}, \hat{\delta})$ and $(Y, d_Y) = (\mathcal{L}, d)$ and to the sequence of maps $g_k := \gamma_k \circ f_k$, where $\hat{S}$ is the one point compactification of S and $\hat{\delta}$ the distance on $\hat{S}$ associated to δ, cf. [4] Lemma 2.2.

Let us denote x_0^k, x_1^k, x_2^k be the points in S such that $f_k(x_i^k) = \theta_i^k$, for $i \in \{0,1,2\}$.

Let us check that the assumptions of lemma 6.2 are verified.

The fact that $lim_{k\to\infty} dist_H(\tilde{D}_k, \hat{S}) = 0$ comes the same way as in [4], (5.5).

By (6.3), we have, $\delta(x_i^k, x_j^k) \geq \frac{\lambda_k}{2} d(\theta_i^k, \theta_j^k)$ and by assumption we then get

$$\delta(x_i^k, x_j^k) \geq \frac{1}{2C}. \tag{7.5}$$

We then get the separation assumption on triples of points by choosing $\delta := inf\{D, \frac{1}{2C}\}$.

It remains to check the assumption on $g_k(\tilde{D}_k) = \gamma_k \circ f_k(\tilde{D}_k) = \gamma_k(D_k)$, namely,

$$lim_{k\to\infty} dist_H(\gamma_k(D_k), \Lambda_{C'}) = 0. \tag{7.6}$$

In order to prove the property (6.6), we want to apply the lemma 6.3, but as the set $(\mathcal{L}, d)$ is a priori not uniformly perfect, we shall replace the uniform perfectness by the fact that $(\mathcal{L}, \lambda_k d, \theta_0)$ converges to a space $(S, \delta_0, 0)$, which is uniformly perfect, cf. ().

We will show the

Lemma 7.4. *We consider the subsets $\tilde{D}_k \subset B_S(0,k)$ and $D_k \subset B_{\lambda_k\mathcal{L}}(\theta_0,k)$, where $\tilde{D}_k$ and D_k are $1/k$-dense subsets of $B_S(0,k)$ and $B_{(\mathcal{L},\lambda_k d)}(\theta_0,k)$, and the bijections $f_k : \tilde{D}_k \to D_k$ coming from the convergence of the sequence of pointed metric spaces $(\mathcal{L}, \lambda_k d, \theta_0)$ to $(S, \delta, 0)$ where $(S, \delta, 0)$ is a weak tangent of $(\partial\tilde{X}, d)$. We also assume that there exist positive constants C and δ, a* *sequence of points* $\theta_1^k, \theta_2^k \in \Lambda_{C'}$ *and a sequence of elements* $\gamma_k \in C'$ *such that* $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C\lambda_k^{-1}$ *and* $d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \delta$ *for all* $0 \leq i \neq j \leq 2$*. Then, the Hausdorf distance* $dist_H(\gamma_k D_k, \mathcal{L})$ *tends to* 0 *as* k *tends to infinity.*

Proof : The proof is word by word the same as the proof of lemma 6.3, ie. lemma 5.1 (i) of [4] with a difference in case 2).

We have $B_{\lambda_k\mathcal{L}}(\theta_0, k) = B_{\mathcal{L}}(\theta_0, \frac{k}{\lambda_k})$ and $D_k \subset B_{\lambda_k\mathcal{L}}(\theta_0, k)$ an $\frac{1}{k}$-dense subset, for the metric $\lambda_k d$. In term of the distance d, the set D_k is $(\epsilon_k R_k)$-dense in $B_{\mathcal{L}}(\theta_0, R_k)$, where $R_k := \frac{k}{\lambda_k}$ and $\epsilon_k := \frac{1}{k^2}$. By assumption, the points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k$ belong to $B_{\mathcal{L}}(\theta_0, \mu_k R_k)$, and satisfy

$$d(\theta_i^k, \theta_j^k) \geq \delta_k R_k \tag{7.7}$$

where $\delta_k := \frac{1}{Ck}$, and $\mu_k := \frac{C}{k}$.

The points $\gamma_k\theta_i^k$ satisfy

$$d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \delta, \tag{7.8}$$

and $\frac{\epsilon_k}{\delta_k^2} = C^2$ is bounded.

Let us consider a point $\theta \in \mathcal{L}$. We want to approximate it by a point of $\gamma_k D_k$.

We can write $\theta = \gamma_k\theta_k$, for some $\theta_k \in \Lambda_{C'}$. There are two cases.

Case 1). For infinitely many indices k, $\theta_k \in B_{\mathcal{L}}(\theta_0, R_k)$. We work in that case for these indices k, thus there are points $\theta'_k \in D_k \cap B_{\mathcal{L}}(\theta_0, R_k)$, with $d(\theta_k, \theta'_k) \leq \epsilon_k R_k$.

Since the distance between the θ_i^k's is bounded below by $\delta_k R_k$, we can find at least two of them which we call a_k and b_k, such that

$$d(\theta_k, b_k) \geq \frac{\delta_k R_k}{2}$$

and,

$$d(\theta'_k, a_k) \geq \frac{\delta_k R_k}{2}. \tag{7.9}$$

As C' is contained in the cocompact group Γ, it acts in a quasi-Möbius way on $(\partial\tilde{X}, d)$

thus,

$$\frac{d(\gamma_k\theta'_k, \gamma_k\theta_k)d(\gamma_k a_k, \gamma_k b_k)}{d(\gamma_k\theta'_k, \gamma_k b_k)d(\gamma_k a_k, \gamma_k\theta_k)} \leq \eta(\frac{d(\theta'_k, \theta_k)d(a_k, b_k)}{d(\theta'_k, b_k)d(\theta_k, a_k)}) \tag{7.10}$$

for some homeomorphism $\eta : [0, \infty) \to [0, \infty)$. This implies

$$d(\gamma_k\theta'_k, \gamma_k\theta_k) \leq \frac{(diam\mathcal{L})^2\eta(8\epsilon_k\mu_k/\delta_k^2)}{\delta}, \tag{7.11}$$

therefore $d(\gamma_k\theta'_k, \gamma_k\theta_k)$ tends to zero as k tends to infinity.

Case 2) . For all but finitely many indices k, $\theta_k \notin B_{\mathcal{L}}(\theta_0, R_k)$.

We work with these indices k such that $\theta_k \notin B_{\mathcal{L}}(\theta_0, R_k)$.

We know that ϵ_k/δ_k^2 is bounded above independantly of k, and by assumption, $\delta_k \leq 2\mu_k$.

We claim that there exist $\xi_k \in D_k$ and a positive constant c_0 such that for all k,

$$\frac{d(\xi_k, \theta_0)}{R_k} \geq c_0 \tag{7.12}$$

let us prove the claim.

On one hand, as $(\partial\tilde{X}, d)$ is uniformly perfect, and so is it's weak tangent (S, δ) because the one point compactification $(\hat{S}, \hat{\delta})$ of (S, δ) is quasi-Möbius homeomorphic to $(\partial\tilde{X}, d)$, therefore there exist a constant $C_0 \in [0, 1)$ such that for every $x \in S$ and $0 < R < diamS$, we have

$$\bar{B}_{(S,\delta)}(0, R) - B_{(S,\delta)}(0, C_0 R) \neq \emptyset. \tag{7.13}$$

On the other hand, $(\mathcal{L}, \lambda_k d, \theta_0)$ converges to $(S, \delta, 0)$. After reindexing the sequence $\{\lambda_k\}$, we have for each $\epsilon > 0$ a map $g_k : B_{\lambda_k \mathcal{L}}(\theta_0, k) \to S$ such that

(i) $g_k(\theta_0) = 0$,

for any two points θ and θ' in $B_{\lambda_k \mathcal{L}}(\theta_0, k)$,

(ii) $|\delta(g_k(\theta), g_k(\theta')) - \lambda_k d(\theta, \theta')| \leq \epsilon$,

(iii) the ϵ-neighborhood of $g_k(B_{\lambda_k \mathcal{L}}(\theta_0, k))$ contains $B_{(S,\delta)}(0, k - \epsilon)$.

By (iii), we have

$$\bar{B}_{(S,\delta)}(0, k - \epsilon) \subset \mathcal{U}_\epsilon^{(S,\delta)} g_k(\bar{B}_{\lambda_k \mathcal{L}}(\theta_0, k)). \tag{7.14}$$

By (6.13) there exist $y_k \in \bar{B}_{(S,\delta)}(0, k - \epsilon) - B_{(S,\delta)}(0, C_0(k - \epsilon))$, and by (6.14) there exist $\xi'_k \in \bar{B}_{\lambda_k \mathcal{L}}(\theta_0, k)$ such that

$$\delta(y_k, g_k(\xi'_k)) \leq \epsilon. \tag{7.15}$$

We now evaluate $d(y_k, g_k(\xi'_k))$. By the above properties (i), (ii), (6.15) and the triangle inequality we have

$$\lambda_k d(\xi'_k, \theta_0) \geq \delta(g_k(\xi'_k), 0) - \epsilon \geq \delta(y_k, 0) - \delta(y_k, g_k(\xi'_k)) - \epsilon$$

$$\geq C_0(k - \epsilon) - 2\epsilon. \tag{7.16}$$

As D_k is $\epsilon_k R_k$-dense in $B_{(\mathcal{L},d)}(\theta_0, k/\lambda_k)$, there exist $\xi_k \in D_k$ such that $d(\xi_k, \xi'_k) \leq \epsilon_k R_k = \frac{k\epsilon_k}{\lambda_k}$.

Let us denote $c_0 = C_0/2$. For k large enough we have $\frac{C_0(k-\epsilon)-2\epsilon-k\epsilon_k}{\lambda_k} \geq \frac{c_0 k}{\lambda_k}$, therefore by (6.16) we get

$$d(\xi_k, \theta_0) \geq d(\xi_k', \theta_0) - d(\xi_k', \xi_k) \geq c_0 R_k, \tag{7.17}$$

which proves the claim.

We can assume that for k large enough, $\mu_k < c_0/2 < 1/2$.

We choose $a_k = \theta_1^k$ and $b_k = \theta_2^k$, and we get

$$\begin{aligned}\frac{d(\gamma_k\xi_k, \gamma_k\theta_k)d(\gamma_k a_k, \gamma_k b_k)}{d(\gamma_k\xi_k, \gamma_k b_k)d(\gamma_k a_k, \gamma_k\theta_k)} &\leq \eta(\frac{d(\xi_k, \theta_k)d(a_k, b_k)}{d(\xi_k, b_k)d(\theta_k, a_k)}) \\ &\leq \eta(\frac{4\mu_k d(\theta_k, \theta_0^k)}{(d(\theta_k, \theta_0^k) - \mu_k R_k))(c_0 - \mu_k)}) \\ &\leq \eta(16\mu_k/c_0). \qquad (7.18)\end{aligned}$$

We get

$$d(\gamma_k\theta_k, \gamma_k\xi_k) \leq (diam\Lambda_{C'})^2 \eta(16\mu_k/c_0)/\delta.$$

□

References

[1] G. Besson, G. Courtois, S. Gallot, Lemme de Schwarz réel et applications géométriques, Acta Mathematica, 183, (1999), 145-169.

[2] G. Besson, G. Courtois, S. Gallot, Hyperbolic manifolds, amalgamated products and crtical exponents., C.R.MathAcad.Sci.Paris, 336, (2003), no. 3, 257-261.

[3] G. Besson, G. Courtois, S. Gallot,Entropies et rigidités des espaces localement symétriques de courbure négative, Geom. Funct. Anal. 5 (1995),731-799.

[4] M. Bonk, B. Kleiner, Rigidity for quasi-Möbius group actions, J. of Diff. Geometry 61, vol. 1, (2002), 81-106.

[5] R. Bott, L.W. Tu, Differential forms in algebraic topology, Springer-Verlag, New-York, (1982).

[6] K. S. Brown, Cohomology of groups, Springer-Verlag, (1982).

[7] D. Burago, Y. Burago, S. Ivanov, A course in Metric Geometry, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, (2001).

[8] G. Carron, E. Pedon, On the differential form spectrum of hyperbolic manifold, to appear.

[9] M. Gromov, Filling riemannian manifolds, J. of Diff. Geometry 18, (1983), 1-147.

[10] V. Kaimanovich, Invariant measures of the geodesic flow and measure at infinity of negatively curved manifolds, Ann. Inst. H.Poincaré Phys. Theor. 53, 4, 361-393, (1994).

[11] A. Lubotsky, Free quotients and the first Betti number of some hyperbolic manifolds, Transformation groups vol. 1 (no 1, 2), (1996), 71-82.

[12] F. Morgan, Geometric measure theory, Academic Press, 1988.

[13] T.Roblin, Ergodicité et équidistribution en courbure négative, Mémoires de la Soc. Math. de France 95, 2003.

[14] J. P. Serre, Arbres, amalgames, SL_2, Société mathématique de France, Astérisque 46, (1977).

[15] Y. Shalom, Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group, Annals of Mathematics 152, (2000), 113-182.

[16] Z.I. Szabó, A short topological proof for the symmetry of @ point homogeneous spaces, Invent. Math. 106, 1991, 61-64.

[17] D. Sullivan Related aspects of positivity in riemannian geometry, J. Differetial Geometry 25, 1987, 327-351.

[18] D. Sullivan, Entropy, hausdorff measure old and new and limit sets of geometrically finite Kleinian groups, Acta Math. 153 (1984) 259-277.

[19] J. Väisälä, Quasi-Möbius maps, Journal d'Analyse Mathématique, vol. 44, 218-234, (1984/85).
[20] C. Yue, Then ergodic theory of discrete isometry groups on manifolds of negative curvature, Trans. Amer. Math. Soc. 348, (1996), 4965-5005.
[21] A. Zeghib, Laminations et hypersurfaces géodésiques des variétés hyperboliques, Ann. Sci. Ecole Norm. Sup. (4), no 2, 1991, 171-188.

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